Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point, II -- Its relevance to the Mathieu equation and the Legendre equation

Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point, II —– its relevance to the Mathieu equation and the Legendre equation

Shingo Kamimoto[a,1], Takahiro Kawai[a,2], Yoshitsugu Takei[a,3]

[a] *Research Institute for Mathematical Sciences Kyoto University Kyoto, 606-8502 JAPAN*

Abstract

We develop the exact WKB analysis of an M2P1T (merging two simple poles and one simple turning point) Schrödinger equation. In Part II, using a WKB-theoretic transformation to the algebraic Mathieu equation constructed in Part I, we calculate the alien derivative of its Borel transformed WKB solutions at each fixed singular point relevant to the simple poles through the analysis of Borel transformed WKB solutions of the Legendre equations. In the course of the calculation of the alien derivative we make full use of microdifferential operators whose symbols are given by the infinite series that appear in the coefficients of the algebraic Mathieu equation and the Legendre equation.

Keywords: exact WKB analysis, M2P1T operator, Mathieu equation, Legendre equation, alien derivative, microdifferential operator, exponential calculus

[1]The research of the authors has been supported in part by JSPS grants-in-aid No.22-6971.

[2]The research of the authors has been supported in part by JSPS grants-in-aid No.20340028.

[3]The research of the authors has been supported in part by JSPS grants-in-aid No.21340029.

0. Introduction

As is mentioned in Part I, our purpose is to develop the exact WKB analysis of an M2P1T equation, that is, a Schrödinger equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q\right)\psi = 0 \quad (\eta : \text{a large parameter}) \tag{0.1}$$

when the potential Q contains merging two simple poles and one turning point. The primary aim is to study the analytic structure of its Borel transformed WKB solutions near fixed singular points relevant to the two simple poles contained in the potential Q. In Part I, we have shown that an M2P1T equation can be transformed to the Mathieu equation. In this Part II, using the transformation to the Mathieu equation constructed in Part I, we calculate the alien derivative of Borel transformed WKB solutions of an M2P1T equation at each fixed singular point relevant to the two simple poles. Note that the Mathieu equation discussed in Part I contains two infinite series in its coefficients. The appearance of these infinite series connotes the necessity of employing microdifferential operators and the growth order conditions on them given in Part I guarantee the existence of such microdifferential operators.

In addition to the transformation constructed in Part I, to separate out the simple turning point of the Mathieu equation from the simple poles so that we may make use of the results of Koike ([Ko]), we consider a WKB-theoretic reduction of the Mathieu equation to the ∞-Legendre equation

$$\left(\frac{d^2}{dx^2} - \eta^2\left(\frac{a\Gamma(a,A,B,\eta)}{x^2-a^2} + \eta^{-2}\left(\frac{\gamma_+(a)}{(x-a)^2} + \frac{\gamma_-(a)}{(x+a)^2}\right)\right)\right)\psi = 0, \tag{0.2}$$

where

(0.3) $\Gamma(a,A,B,\eta) = \sum_k \Gamma_k(a,A,B)\eta^{-k}$ with $\Gamma_k(a,A,B)$ satisfying appropriate growth order conditions

and

(0.4) $\gamma_\pm(a)$ are holomorphic functions at $a=0$,

near two simple poles and we let the parameter B contained in the Mathieu equation sufficiently small. (Note that the parameter B corresponds to the parameter ρ contained in the potential of an M2P1T equation; see [Part I, Definition 1.1].) As its consequence, we will obtain our main theorem (Theorem 4.2 in Section 4), which explicitly describes the alien derivative of Borel transformed WKB solutions of an M2P1T equation at each fixed singular point relevant to merging two simple poles.

The main results in this article were announced in [KKT].

Acknowledgment.

We sincerely thank Professor T. Koike for providing us with his draft concerning the Voros coefficients of the Legendre equation.

1. Reduction of the Mathieu equation to the Legendre equation near its simple poles

The main purpose of this section is to construct a transformation that brings the Mathieu equation

$$\left(\frac{d^2}{dx^2}-\eta^2\left(\frac{aA+xB}{x^2-a^2}+\eta^{-2}\left(\frac{g_+(a)}{(x-a)^2}+\frac{g_-(-a)}{(x+a)^2}\right)\right)\right)\tilde{\psi}=0 \tag{1.1}$$

with genuine constants $A(\neq 0)$ and B to the following Legendre equation

$$\left(\frac{d^2}{dz^2}-\eta^2\left(\frac{a\Lambda^2}{z^2-a^2}+\eta^{-1}\frac{\sqrt{a}\Lambda}{z^2-a^2}+\eta^{-2}\frac{az\nu+a^2(\mu^2-1)}{(z^2-a^2)^2}\right)\right)\phi=0 \tag{1.2}$$

on a neighborhood of the line segment connecting two simple poles at $x=\pm a$.

We note that introducing the large parameter η as in (1.2) to the classical Legendre equation is a natural one from the WKB-theoretic viewpoint; an elementary evidence for the naturalness is given by the fact that the WKB solutions $\psi_\pm$ of (1.2) with $\nu=0$ and $\mu^2=1/4$ is expressed in a closed form, i.e.,

$$\left(\eta\sqrt{a}\Lambda+\frac{1}{2}\right)^{-1/2}(z^2-a^2)^{1/4}\left(\frac{z+\sqrt{z^2-a^2}}{a}\right)^{\pm(\eta\sqrt{a}\Lambda+1/2)}, \tag{1.3}$$

which forms a counterpart of the interesting formula for $P^{\pm 1/2}_{\sqrt{a}\Lambda}$ and $Q^{\pm 1/2}_{\sqrt{a}\Lambda}$ ([Er, vol.I, p.150])). This naturalness seems to have enabled Koike ([Ko]) to find the explicit form of the Voros coefficient for (1.2), of which we will make essential use in Section 3. However, there is one technical problem with the equation (1.2); it contains a term with degree 1 in η. Although the appearance of degree 1 (or, more generally, an odd degree part) in η is natural from the viewpoint of the general theory of simple-pole type operators (cf. e.g. [KKoT]), it is somewhat unhandy in this paper; the equations we are dealing with in this paper contain only even degree terms in η. Hence, as an auxiliary equation we consider the following equation:

$$\left(\frac{d^2}{dz^2} - \eta^2\left(\frac{a\Gamma}{z^2-a^2} + \eta^{-2}\left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2}\right)\right)\right)\psi = 0, \tag{1.4}$$

which can be smoothly related with (1.1). We will show the WKB-theoretic equivalence of (1.2) and (1.4) later in Proposition 1.1. Thus our first task is to construct the transformation series

$$z(x, a, A, B, \eta) = \sum_{n=0}^{\infty} z_{2n}(x, a, A, B)\eta^{-2n} \tag{1.5}$$

and

$$\Gamma(a, A, B, \eta) = \sum_{n=0}^{\infty} \Gamma_{2n}(a, A, B)\eta^{-2n} \tag{1.6}$$

so that they satisfy

$$\begin{aligned} &\frac{aA+xB}{x^2-a^2} + \eta^{-2}\left(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2}\right) \\ &= \left(\frac{\partial z}{\partial x}\right)^2 \left(\frac{a\Gamma}{z^2-a^2} + \eta^{-2}\left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2}\right)\right) \\ &\quad - \frac{1}{2}\eta^{-2}\{z; x\}. \end{aligned} \tag{1.7}$$

In order to attain the required reduction of the Mathieu equation to the Legendre equation near its simple poles, we need several delicate properties of the series including their domains of definition and estimates. Hence the precise target is to prove the following

Theorem 1.1. *There exist holomorphic functions $z_{2n}(x, a, A, B)$ and $\Gamma_{2n}(a, A, B)$ on*

$$E^2_{r_1,r_2} = \left\{(x, a, A, B) \in \mathbb{C}^4 : |x| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\right\} \tag{1.8}$$

for some constants $r_1 > 1$ *and* $r_2 > 0$ *such that* $z(x, a, A, B, \eta)$ *and* $\Gamma(a, A, B, \eta)$ *respectively given by* (1.5) *and* (1.6) *satisfy* (1.7) *and the following conditions there:*

(1.9) *the function* $z_0(x, a, A, B)$ *of* x *is injective on* $D_{r_1|a|} = \{x \in \mathbb{C} : |x| < r_1|a|\}$ *for fixed* a, A *and* B,

$$z_0(\pm a, a, A, B) = \pm a, \tag{1.10}$$

$$\frac{\partial z_0}{\partial x}(x, a, A, B) \neq 0. \tag{1.11}$$

Furthermore they satisfy the following estimates: for any $h > 0$ *we can take sufficiently small* $\delta > 0$ *so that*

$$|z_{2n}(x, a, A, B)| \leq (2n)!h^n|aA|^{-n}, \tag{1.12}$$

$$|\Gamma_{2n}(a, A, B)| \leq (2n)!h^n|aA|^{-n} \tag{1.13}$$

hold on $E^2_{r_1,\delta}$ *for* $n \geq 1$.

In order to explain the geometric meaning of Theorem 1.1, we give some remarks before beginning its proof.

Remark 1.1. Since the two simple poles of (1.1) are contained in $D_{r_1|a|}$, Theorem 1.1 guarantees that the reduction of (1.1) to (1.4) is successful on a full neighborhood of the line segment joining these two poles. On the other hand, Theorem 1.1 does not say anything about the simple turning point of (1.1).

Remark 1.2. Two simple poles at $x = \pm a$ and the simple turning point at $x = -aA/B$ all merge at the origin when a tends to 0. But, by taking B/A sufficiently small, we can regard that the turning point is sufficiently far away from the two simple poles in the scale of a.

Proof of Theorem 1.1. Let $\tilde{x}, \tilde{z}, \tilde{B}$ and $\tilde{\eta}$ be

$$\tilde{x} = x/a, \tag{1.14}$$

$$\tilde{z} = z/a, \tag{1.15}$$

$$\tilde{B} = B/A, \tag{1.16}$$

$$\tilde{\eta} = \sqrt{aA}\eta, \tag{1.17}$$

then (1.1) is rewritten as follows:

$$\left(\frac{d^2}{d\tilde{x}^2} - \tilde{\eta}^2\left(\frac{1+\tilde{x}\tilde{B}}{\tilde{x}^2-1} + \tilde{\eta}^{-2}\left(\frac{g_+(a)}{(\tilde{x}-1)^2} + \frac{g_-(-a)}{(\tilde{x}+1)^2}\right)\right)\right)\tilde{\psi} = 0. \tag{1.18}$$

Hence if we construct

$$\tilde{z}(\tilde{x}, \tilde{B}, \tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{z}_{2n}(\tilde{x}, \tilde{B})\tilde{\eta}^{-2n} \tag{1.19}$$

and

$$\tilde{\Gamma}(\tilde{B}, \tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{\Gamma}_{2n}(\tilde{B})\tilde{\eta}^{-2n} \tag{1.20}$$

so that they satisfy

$$\begin{aligned} &\frac{1+\tilde{x}\tilde{B}}{\tilde{x}^2-1} + \tilde{\eta}^{-2}\left(\frac{g_+(a)}{(\tilde{x}-1)^2} + \frac{g_-(-a)}{(\tilde{x}+1)^2}\right) \\ &= \left(\frac{\partial \tilde{z}}{\partial \tilde{x}}\right)^2 \left(\frac{\tilde{\Gamma}}{\tilde{z}^2-1} + \tilde{\eta}^{-2}\left(\frac{g_+(a)}{(\tilde{z}-1)^2} + \frac{g_-(-a)}{(\tilde{z}+1)^2}\right)\right) \\ &\qquad - \frac{1}{2}\tilde{\eta}^{-2}\{\tilde{z}; \tilde{x}\}, \end{aligned} \tag{1.21}$$

then we find

$$z(x, a, A, B, \eta) = a\tilde{z}(x/a, B/A, \sqrt{aA}\eta) \tag{1.22}$$

and

$$\Gamma(a, A, B, \eta) = A\tilde{\Gamma}(B/A, \sqrt{aA}\eta) \tag{1.23}$$

satisfy (1.7).

Therefore it suffices to show the following properties of $\tilde{z}$ and $\tilde{\Gamma}$:

(1.24) $\tilde{z}_{2n}(\tilde{x}, \tilde{B})$ and $\tilde{\Gamma}_{2n}(\tilde{B})$ are holomorphic on $\tilde{E}^2_{r_1,r_2} = \left\{(\tilde{x}, \tilde{B}) \in \mathbb{C}^2 : |\tilde{x}| \le r_1, |\tilde{B}| \le r_2\right\}$ for some positive constants $r_1 > 1$ and $r_2 > 0$, (1.25) the function $\tilde{z}_0(\tilde{x}, \tilde{B})$ of $\tilde{x}$ is injective on $D_{r_1} = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| \leq r_1\}$ for fixed $\tilde{B}$ with $|\tilde{B}| \leq r_2$,

$$\tilde{z}_0(\pm 1, \tilde{B}) = \pm 1, \tag{1.26}$$

$$\frac{\partial \tilde{z}_0}{\partial \tilde{x}}(\tilde{x}, \tilde{B}) \neq 0 \quad \text{on} \quad D_{r_1} \tag{1.27}$$

and they satisfy the following estimates: for any $h > 0$ we can take sufficiently small $\delta > 0$ so that

$$|\tilde{\Gamma}_{2n}(\tilde{B})| \leq (2n)! h^n, \tag{1.28}$$

$$|\tilde{z}_{2n}(\tilde{x}, \tilde{B})| \leq (2n)! h^n \tag{1.29}$$

hold on $\tilde{E}^2_{r_1,\delta}$ for $n \geq 1$.

We first show (1.25), (1.26) and (1.27). Comparing the coefficients of $\tilde{\eta}^0$ of (1.21), we find that $\tilde{z}_0$ and $\tilde{\Gamma}_0$ satisfy

$$\frac{1 + \tilde{x}\tilde{B}}{1 - \tilde{x}^2} = \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \frac{\tilde{\Gamma}_0}{1 - \tilde{z}_0^2}. \tag{1.30}$$

Therefore we take $\tilde{z}_0$ and $\tilde{\Gamma}_0$ as follows:

$$\tilde{z}_0(\tilde{x}, \tilde{B}) = \cos\left(\frac{1}{\sqrt{\tilde{\Gamma}_0}} \int_1^{\tilde{x}} \sqrt{\frac{1 + \tilde{x}\tilde{B}}{1 - \tilde{x}^2}} d\tilde{x}\right), \tag{1.31}$$

$$\sqrt{\tilde{\Gamma}_0(\tilde{B})} = \frac{-1}{\pi} \int_1^{-1} \sqrt{\frac{1 + \tilde{x}\tilde{B}}{1 - \tilde{x}^2}} d\tilde{x}. \tag{1.32}$$

From (1.31) and (1.32), we immediately find that

$$\tilde{z}_0(\tilde{x}, 0) = \tilde{x}, \tag{1.33}$$

$$\sqrt{\tilde{\Gamma}_0(0)} = 1 \tag{1.34}$$

and (1.26) hold. Let $r_1 > 1$ be a constant. Then, for any positive constant ε, we can take $\delta > 0$ so that $\tilde{z}_0(\tilde{x}, \tilde{B})$ and $\tilde{\Gamma}_0(\tilde{B})$ are holomorphic on $\tilde{E}^2_{r_1+2\varepsilon,\delta} = \{(\tilde{x}, \tilde{B}) \in \mathbb{C}^2 : |\tilde{x}| \le r_1 + 2\varepsilon, |\tilde{B}| \le \delta\}$ and satisfy the following estimates there:

$$\max\left\{|\tilde{z}_0(\tilde{x}, \tilde{B}) - \tilde{x}|, \left|\frac{\partial \tilde{z}_0}{\partial \tilde{x}} - 1\right|, |\tilde{\Gamma}_0(\tilde{B}) - 1|\right\} < \varepsilon. \tag{1.35}$$

Therefore, for any $y \in \tilde{z}_0(\tilde{E}^2_{r_1,\delta})$, we find $|\tilde{x} - y| > \varepsilon$ holds on $\{\tilde{x} \in \mathbb{C} : |\tilde{x}| = r_1 + 2\varepsilon\}$ and hence, appealing to Rouché's theorem, we find that (1.25) holds for $r_2 < \delta$. Note that (1.27) follows also from (1.35).

Next we show (1.24). Comparing the coefficients of $\tilde{\eta}^{-2n}$ $(n \ge 1)$ of (1.21), we obtain the following relations for $\left(\tilde{z}_{2n}, \tilde{\Gamma}_{2n}\right)$ $(n \ge 1)$:

$$\frac{2\tilde{\Gamma}_0}{\tilde{z}_0^2 - 1}\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} - \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \frac{2\tilde{z}_0\tilde{\Gamma}_0}{(\tilde{z}_0^2 - 1)^2}\tilde{z}_{2n} + \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \frac{\tilde{\Gamma}_{2n}}{\tilde{z}_0^2 - 1} = \tilde{\Phi}_{2n}, \tag{1.36}$$

where $\tilde{\Phi}_{2n}$ $(n \ge 1)$ is a sum of terms that are determined by $\left(\tilde{z}_{2k}, \tilde{\Gamma}_{2k}\right)$ $(0 \le k \le n-1)$. Multiplying both sides of (1.36) by $(\tilde{z}_0^2 - 1)/(2\tilde{\Gamma}_0 \partial\tilde{z}_0/\partial\tilde{x})$, we can rewrite (1.36) as follows:

$$\frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} - \frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\tilde{z}_0}{\tilde{z}_0^2 - 1}\tilde{z}_{2n} + \frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0} = \Phi_{2n}. \tag{1.37}$$

Now, we give the concrete form of Φ_{2n} $(n \ge 1)$. The concrete form of Φ_2 is rather simple:

$$\begin{aligned}\Phi_2 = \frac{\tilde{z}_0^2 - 1}{2\tilde{\Gamma}_0}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1} &\left\{\frac{1}{2}\{\tilde{z}_0; \tilde{x}\} + \left(\frac{g_+(a)}{(\tilde{x} - 1)^2} + \frac{g_-(-a)}{(\tilde{x} + 1)^2}\right)\right.\\ &\left. - \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \left(\frac{g_+(a)}{(\tilde{z}_0 - 1)^2} + \frac{g_-(-a)}{(\tilde{z}_0 + 1)^2}\right)\right\}.\end{aligned} \tag{1.38}$$

Then, to simplify the expression of Φ_{2n} $(n \ge 2)$ and also the discussion given below, we introduce $y_{2k}(\tilde{x}, \tilde{B})$ $(k = 0, 1, \cdots)$ by

$$y_0(\tilde{x}, \tilde{B}) = \frac{\tilde{z}_0^2(\tilde{x}, \tilde{B}) - 1}{\tilde{x}^2 - 1}, \tag{1.39}$$

$$y_{2k}(\tilde{x},\tilde{B}) = \frac{1}{\tilde{x}^2-1}\sum_{l=0}^{k}\tilde{z}_{2l}(\tilde{x},\tilde{B})\tilde{z}_{2(k-l)}(\tilde{x},\tilde{B}) \quad (k \geq 1). \tag{1.40}$$

We immediately see that they satisfy the following relation:

$$(\tilde{x}^2-1)\sum_{n=0}^{\infty}\tilde{\eta}^{-2n}y_{2n}(\tilde{x},\tilde{B}) = \tilde{z}^2(\tilde{x},\tilde{B},\tilde{\eta}) - 1. \tag{1.41}$$

Further we use the following notation: for a multi-index $\vec{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_\mu)$ in $\mathbb{N}_0^\mu$ and for κ_j-dependent $(j = 1, 2, \cdots, \mu)$ quantities X_{κ_j}, we define

$$|\vec{\kappa}|_\mu = \sum_{j=1}^{\mu}\kappa_j, \tag{1.42}$$

$$\sum_{|\vec{\kappa}|_\mu = k}^{*} X_{\kappa_1}\cdots X_{\kappa_\mu} = \begin{cases} 1 & \text{for } \mu = 0 \\ \displaystyle\sum_{\substack{|\vec{\kappa}|_\mu = k \\ \kappa_j \geq 1}} X_{\kappa_1}\cdots X_{\kappa_\mu} & \text{for } \mu \geq 1. \end{cases} \tag{1.43}$$

Using these notations, we can describe the concrete form of Φ_{2n} $(n \geq 2)$ as follows:

$$\Phi_{2n} = \Phi_{2n}^{(1)} + \Phi_{2n}^{(2)} + \Phi_{2n}^{(3)}, \tag{1.44}$$

where $\Phi_{2n}^{(1)}$, $\Phi_{2n}^{(2)}$ and $\Phi_{2n}^{(3)}$ are

$$\begin{aligned}\Phi_{2n}^{(1)} = &-\frac{1}{2}\frac{d\tilde{z}_0}{d\tilde{x}}\sum_{\mu=2}^{n}\sum_{|\vec{\kappa}|_\mu = n}^{*}(-1)^\mu\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu} \\ &-\frac{1}{2\tilde{\Gamma}_0}\sum_{\substack{k_1+\cdots+k_4=n \\ k_1,\cdots,k_4\leq n-1}}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\frac{d\tilde{z}_{2k_2}}{d\tilde{x}}\tilde{\Gamma}_{2k_3} \\ &\times\sum_{\mu=\min\{1,k_4\}}^{k_4}\sum_{|\vec{\kappa}|_\mu = k_4}^{*}(-1)^\mu\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu} \\ &+\frac{1}{2}\frac{1}{\tilde{z}_0^2-1}\frac{d\tilde{z}_0}{d\tilde{x}}\sum_{k_1+k_2=n}^{*}\tilde{z}_{2k_1}\tilde{z}_{2k_2},\end{aligned} \tag{1.45}$$

$$
\begin{aligned}
(1.46)\quad \Phi_{2n}^{(2)} =& \frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\sum_{k_1+k_2=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\frac{d^3\tilde{z}_{2k_1}}{d\tilde{x}^3}\\
&\times\sum_{\mu=\min\{1,k_2\}}^{k_2}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\\
&-\frac{3(\tilde{z}_0^2-1)}{8\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3}\frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2}\frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2}\\
&\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}},\\
(1.47)\quad \Phi_{2n}^{(3)} =& -\frac{\tilde{z}_0^2-1}{2\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\frac{d\tilde{z}_{2k_2}}{d\tilde{x}}\frac{g_+(a)}{(\tilde{z}_0-1)^2}\\
&\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\frac{\tilde{z}_{2\kappa_1}\cdots\tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0-1)^\mu}\\
&-\frac{\tilde{z}_0^2-1}{2\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\frac{d\tilde{z}_{2k_2}}{d\tilde{x}}\frac{g_-(-a)}{(\tilde{z}_0+1)^2}\\
&\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\frac{\tilde{z}_{2\kappa_1}\cdots\tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0+1)^\mu}.
\end{aligned}
$$

Then we recursively determine $(\tilde{z}_{2n}(\tilde{x},\tilde{B}),\tilde{\Gamma}_{2n}(\tilde{B}))$ $(n\geq 1)$ as follows:

$$
(1.48)\qquad \tilde{\Gamma}_{2n}(\tilde{B}) = \frac{-2\tilde{\Gamma}_0}{\pi}\int_1^{-1}(1-\tilde{z}_0^2)^{-1/2}\Phi_{2n}(\tilde{x},\tilde{B})d\tilde{x},
$$

$$
\begin{aligned}
(1.49)\qquad \tilde{z}_{2n}(\tilde{x},\tilde{B}) &= (1-\tilde{z}_0^2)^{1/2}\int_1^{\tilde{x}}(1-\tilde{z}_0^2)^{-1/2}\left(\Phi_{2n}(\tilde{x},\tilde{B})-\frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0}\frac{d\tilde{z}_0}{d\tilde{x}}\right)d\tilde{x}\\
&= (1-\tilde{z}_0^2)^{1/2}\int_{-1}^{\tilde{x}}(1-\tilde{z}_0^2)^{-1/2}\left(\Phi_{2n}(\tilde{x},\tilde{B})-\frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0}\frac{d\tilde{z}_0}{d\tilde{x}}\right)d\tilde{x}.
\end{aligned}
$$

Now we inductively confirm that $(\tilde{z}_{2n}(\tilde{x},\tilde{B}),\tilde{\Gamma}_{2n}(\tilde{B}))$ $(n\geq 1)$ satisfy (1.24), (1.37) and

$$
(1.50)\qquad \tilde{z}_{2n}(\pm 1,\tilde{B}) = 0.
$$

We first confirm that $(\tilde{z}_2(\tilde{x},\tilde{B}),\tilde{\Gamma}_2(\tilde{B}))$ satisfies (1.24) and (1.50). From (1.27) we immediately see that $\{\tilde{z}_0;\tilde{x}\}$ is holomorphic on $\tilde{E}^2_{r_1,r_2}$. Furthermore, using (1.25) and (1.26), we find that

$$
\begin{aligned}
(1.51)\qquad & (\tilde{z}_0^2-1)\left(\frac{g_+(a)}{(\tilde{x}-1)^2}-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\frac{g_+(a)}{(\tilde{z}_0-1)^2}\right)\\
&=g_+(a)\frac{\tilde{z}_0+1}{\tilde{z}_0-1}\left(\left(\frac{\tilde{z}_0-1}{\tilde{x}-1}\right)^2-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\right)
\end{aligned}
$$

is holomorphic at $\tilde{x}=1$ and hence on $\tilde{E}^2_{r_1,r_2}$. By the same reasoning, the counterpart of (1.51) in Φ_2, i.e.,

$$
\begin{aligned}
(1.52)\qquad & (\tilde{z}_0^2-1)\left(\frac{g_-(a)}{(\tilde{x}+1)^2}-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\frac{g_-(a)}{(\tilde{z}_0+1)^2}\right)\\
&=g_-(a)\frac{\tilde{z}_0-1}{\tilde{z}_0+1}\left(\left(\frac{\tilde{z}_0+1}{\tilde{x}+1}\right)^2-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\right)
\end{aligned}
$$

is also holomorphic on $\tilde{E}^2_{r_1,r_2}$. In conclusion Φ_2 is holomorphic on $\tilde{E}^2_{r_1,r_2}$. It is clear from the representation (1.48) (resp. (1.49)) of $\tilde{\Gamma}_2$ (resp. $\tilde{z}_2$) that they are holomorphic and satisfy (1.37) and (1.50) on $\tilde{E}^2_{r_1,r_2}$.

Next we confirm that $(\tilde{z}_{2n},\tilde{\Gamma}_{2n})$ satisfies (1.24), (1.37) and (1.50) under the assumption that $(\tilde{z}_{2k},\tilde{\Gamma}_{2k})$ $(1\le k\le n-1)$ satisfy these properties. By the same reasoning as the case of $n=1$, it suffices to show that Φ_{2n} is holomorphic on $\tilde{E}^2_{r_1,r_2}$. We first note that, since $\tilde{z}_0$ satisfies (1.25) and (1.26), y_0 is holomorphic and satisfies

$$
(1.53)\qquad y_0(\tilde{x},\tilde{B})\neq 0 \quad\text{on}\quad \tilde{E}^2_{r_1,r_2}.
$$

Further the holomorphy of y_{2k} $(1\le k\le n-1)$ follows from the induction hypothesis (1.50). Then the holomorphy of $\Phi^{(1)}_{2n}$ and $\Phi^{(2)}_{2n}$ immediately follows from the induction hypothesis also. On the other hand the seeming poles at $\tilde{x}=\pm a$ that appear in (1.47) are cancelled out thanks to [Part I, Lemma C.1], and hence $\Phi^{(3)}_{2n}$ is holomorphic on $\tilde{E}^2_{r_1,r_2}$. (Indeed, we can apply [Part I, Lemma C.1] with $w_0=\tilde{z}_0\pm 1$ and $w_k=\tilde{z}_{2k}$ $(k=1,2,\cdots)$ in this case.) Thus, we find that Φ_{2n} is holomorphic on $\tilde{E}^2_{r_1,r_2}$. Then the induction proceeds, and hence we obtain (1.24) and (1.50) for $n\ge 1$.

Now we embark on the proof of the estimates (1.28) and (1.29). Let N be an arbitrarily large natural number. In order to derive these estimates, we introduce a new variable ζ given by

$$\zeta = \exp\left[\frac{1}{N}\log\left(\frac{\tilde{x}}{N}\right)\right] \tag{1.54}$$

and we consider a holomorphic function $g(\tilde{x})$ on $D_N = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| \leq N\}$ as a holomorphic function $g(N\zeta^N)$ on $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$.

Remark 1.3. As we will see below, to obtain (1.28) and (1.29) for arbitrarily small h, we will let N sufficiently large so that (1.65) holds. Then D_N becomes larger and larger as N increases. Still, the same reasoning as in the proof of (1.24), (1.25) and (1.27) guarantee that,

(1.55) for arbitrary large N, we can take $\delta > 0$ sufficiently small so that $\tilde{z}_{2n}(\tilde{x}, \tilde{B})$ $(n = 0, 1, 2, \cdots)$ are holomorphic on $\tilde{E}^2_{N,\delta}$.

In what follows, we use the following notation: for a holomorphic function $f(\zeta)$ on $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$, we define $\|f\|_{\{\varepsilon\}}$ by

$$\|f\|_{\{\varepsilon\}} := \sup_{|\zeta|\leq 1-\varepsilon} |f(\zeta)| \tag{1.56}$$

for $0 < \varepsilon < 1$. Then our task is to show the following

Lemma 1.1. *There exist positive constants $C_0(< 1)$ and C_1 such that, for arbitrarily large natural number N, we can take a sufficiently small positive constant δ (depending on N) so that the following estimates hold for $|\tilde{B}| \leq \delta$ and $0 < \varepsilon \leq (2N)^{-1}\log N$: for $1 \leq k \leq N-1$,*

$$|\tilde{\Gamma}_{2k}| \leq C_0 N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k, \tag{1.57}$$

$$\|\tilde{z}_{2k}\|_{\{\varepsilon\}} \leq C_0 N^{k+1-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k, \tag{1.58}$$

$$\left\|\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\varepsilon\}} \leq C_0 N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k, \tag{1.59}$$

$$\left\|\frac{y_{2k}}{y_0}\right\|_{\{\varepsilon\}} \leq C_0 N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k, \tag{1.60}$$

and for $k \geq N$,

$$|\tilde{\Gamma}_{2k}| \leq C_0 N^{-1}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k, \tag{1.61}$$

$$\|\tilde{z}_{2k}\|_{\{\varepsilon\}} \le C_0(\varepsilon N)^{-2k}(2k)!(C_1 \log N)^k, \tag{1.62}$$

$$\left\| \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \le C_0 N^{-1}(\varepsilon N)^{-2k}(2k)!(C_1 \log N)^k, \tag{1.63}$$

$$\left\| \frac{y_{2k}}{y_0} \right\|_{\{\varepsilon\}} \le C_0 N^{-1}(\varepsilon N)^{-2k}(2k)!(C_1 \log N)^k. \tag{1.64}$$

As the proof of Lemma 1.1 is delicate and lengthy, we describe its role in our whole reasoning before proving it; the proof of Lemma 1.1 will be given after we explain its role. Now a crucial point is that the estimate (1.28) (resp. (1.29)) we want to prove follows from (1.57) and (1.61) (resp. (1.58) and (1.62)). This can be confirmed in the following manner: Let $h > 0$ be an arbitrarily small number. Then we take N so that it satisfies

$$\frac{4C_1}{\log N} < h. \tag{1.65}$$

By taking $\varepsilon = (2N)^{-1} \log N$, we obtain the following estimates from (1.57) and (1.61) for $n \ge 1$:

$$|\tilde{\Gamma}_{2n}(\tilde{B})| \le C_0 N^{-1}(2n)!\Big(\frac{4C_1}{\log N}\Big)^n \tag{1.66}$$

for $|\tilde{B}| \le \delta$, where δ is a positive constant appearing in Lemma 1.1. By the same way, we obtain the following estimates from (1.58) and (1.62):

$$|\tilde{z}_{2n}(\tilde{x}, \tilde{B})| = |\tilde{z}_{2n}(N\zeta^N, \tilde{B})| \le C_0(2n)!\Big(\frac{4C_1}{\log N}\Big)^n \tag{1.67}$$

for $|\zeta| \le 1 - (2N)^{-1} \log N$ and $|\tilde{B}| \le \delta$. Here we note that, for sufficiently large N,

$$|\tilde{x}| \ge N^{1/2}/2 \text{ holds when } |\zeta| = 1 - (2N)^{-1} \log N. \tag{1.68}$$

Indeed, (1.68) follows from the relation $\tilde{x} = N\zeta^N$ and the following inequality:

$$N\Big(1 - \frac{\log N}{2N}\Big)^N \ge \frac{1}{2} N \exp\Big[-\frac{1}{2}\log N\Big] = \frac{1}{2}N^{1/2} \tag{1.69}$$

holds for sufficiently large N. Thus we can assume that (1.67) holds for $|\tilde{x}| \le N^{1/2}/2$. Hence, by taking N so that it satisfies $r_1 \le N^{1/2}/2$ and (1.65), we obtain (1.28) and (1.29). In conclusion, we obtain Theorem 1.1. Thus the proof of Theorem 1.1 will be completed if we verify Lemma 1.1.

Proof of Lemma 1.1. To begin with we confirm that (1.57) $\sim$ (1.60) hold for $k=1$. We first show that Φ_2 satisfies the following estimates: there exists a positive constant $\tilde{C}_0$ such that, for an arbitrary positive constant $p>1$, we can take a positive constant δ so that

$$\sup_{|\tilde{x}|\le N} |\Phi_2(\tilde{x},\tilde{B})| \le \tilde{C}_0 N^{-p+1} \tag{1.70}$$

holds for $|\tilde{B}|\le\delta$.

Remark 1.4. As (1.33) and (1.34) indicate, we readily find

$$\Phi_{2n}(\tilde{x},0)=0 \quad \text{for} \quad n\ge 1. \tag{1.71}$$

Therefore it is natural to expect that (1.70) holds by taking δ sufficiently small depending on N and p.

Indeed, by taking $\delta>0$ sufficiently small, we may assume that $\tilde{\Gamma}_0$, y_0 and $d\tilde{z}_0/d\tilde{x}$ are holomorphic on $\tilde{E}^2_{N+N^p,\delta}$. Furthermore, since $\tilde{\Gamma}_0(0)=y_0(\tilde{x},0)=d\tilde{z}_0/d\tilde{x}(\tilde{x},0)=1$, by letting $\delta>0$ sufficiently small again, we may also assume that

$$\sup_{\substack{|\tilde{x}|\le N+N^p\\ |\tilde{B}|\le\delta}} \left\{ |(\tilde{\Gamma}_0)^{\pm1}|, |(y_0)^{\pm1}|, \left| \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{\pm1} \right| \right\} \le 2 \tag{1.72}$$

holds. We fix $\tilde{B}$ in the disc $\{\tilde{B} : |\tilde{B}|\le\delta\}$. Then we obtain the following estimates for $j=1,2,\cdots$ from Cauchy's inequality:

$$\sup_{|\tilde{x}|\le N} \left| \frac{d^j\tilde{z}_0}{d\tilde{x}^j} \right| \le 2(j-1)!N^{-(j-1)p}. \tag{1.73}$$

Therefore, from (1.73) we obtain

$$\begin{aligned} |\{\tilde{z}_0;\tilde{x}\}| &\le \left| \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1} \frac{d^3\tilde{z}_0}{d\tilde{x}^3} \right| + \frac{3}{2} \left| \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2} \left(\frac{d^2\tilde{z}_0}{d\tilde{x}^2}\right)^2 \right| \\ &\le 32N^{-2p} \end{aligned} \tag{1.74}$$

on D_N. In what follows, we fix p at $N+2$ and take δ sufficiently small so that (1.70) holds.

Next we derive the estimates of (1.51) on D_N. Since (1.51) is holomorphic on D_N, appealing to the maximum modulus principle, it suffices to estimate (1.51) on the boundary ∂D_N of D_N. Further, since $g_\pm(x)$ is holomorphic at the origin, we can assume that

$$|g_\pm(\pm a)| \le \tilde{C}_1 \tag{1.75}$$

holds for some positive constant $\tilde{C}_1$. From the representation

$$\tilde{z}_0(\tilde{x}) = 1 + (\tilde{x}-1)\frac{\partial \tilde{z}_0}{\partial \tilde{x}} - \int_1^{\tilde{x}} (\tilde{x}-1)\frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x} \tag{1.76}$$

of $\tilde{z}_0(\tilde{x})$, we obtain

$$\begin{aligned}\left(\frac{\tilde{z}_0-1}{\tilde{x}-1}\right)^2 - \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 = &-\frac{2}{\tilde{x}-1}\frac{\partial \tilde{z}_0}{\partial \tilde{x}} \int_1^{\tilde{x}} (\tilde{x}-1)\frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x} \\ &+ \left(\frac{1}{\tilde{x}-1}\int_1^{\tilde{x}} (\tilde{x}-1)\frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x}\right)^2.\end{aligned} \tag{1.77}$$

Here we note that it follows from (1.73) that the following estimates hold on ∂D_N:

$$\left|\frac{1}{\tilde{x}-1}\int_1^{\tilde{x}} (\tilde{x}-1)\frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x}\right| \le \frac{2(N+1)^2}{N-1} N^{-p}. \tag{1.78}$$

Further, by taking δ sufficiently small, we may assume that (1.35) holds with $\varepsilon = 1/2$ on D_N, and hence,

$$N - \frac{3}{2} \le |\tilde{z}_0 \pm 1| \le N + \frac{3}{2} \tag{1.79}$$

holds on ∂D_N. Then, combining (1.75), (1.78) and (1.79), we obtain the following estimates of (1.51):

$$\begin{aligned}&\sup_{|\tilde{x}|=N} \left| g_+(a)\frac{\tilde{z}_0+1}{\tilde{z}_0-1}\left(\left(\frac{\tilde{z}_0-1}{\tilde{x}-1}\right)^2 - \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2\right)\right| \\ &\le \tilde{C}_1 \frac{N+3/2}{N-3/2}\left(\frac{8(N+1)^2}{N-1}N^{-p} + \left(\frac{2(N+1)^2}{N-1}N^{-p}\right)^2\right)\end{aligned} \tag{1.80}$$

$$\leq 320\tilde{C}_1 N^{-p+1}.$$

By the same reasoning, we obtain the same estimates with (1.80) for (1.52). Therefore, combining (1.72), (1.74), (1.79) and (1.80), we obtain (1.70).

Now we derive (1.57) $\sim$ (1.60) from (1.70) for $k = 1$. Since $0 < \varepsilon \leq (2N)^{-1}\log N$, (1.57) for $k = 1$ immediately follows from the representation (1.48) of Γ_2, (1.70) for $C_1 \geq (C_0)^{-1}\tilde{C}_0$ as follows:

$$\begin{aligned}
(1.81)\qquad |\tilde{\Gamma}_2(\tilde{B})| &\leq 4\Big|\frac{\tilde{\Gamma}_0}{\pi}\Big| \sup_{|\tilde{x}|\leq N}|\Phi_2| \int_1^{-1} |1-\tilde{z}_0^2|^{-1/2}\Big|\Big(\frac{d\tilde{z}_0}{d\tilde{x}}\Big)^{-1}\Big||d\tilde{z}_0| \\
&\leq 8\tilde{C}_0 N^{-p+1} \\
&\leq 8\tilde{C}_0 N^{-p+1}(\varepsilon N)^{-2}2^{-2}(\log N)^2 \\
&\leq C_0 N^{-p+2}(\varepsilon N)^{-2}2!(C_0)^{-1}\tilde{C}_0 \log N.
\end{aligned}$$

Next we consider the estimates of $\tilde{z}_2$. Since $\tilde{z}_2$ is holomorphic on D_N, it suffices to estimate it for $\tilde{x} \in \partial D_N$. We obtain the following estimates from (1.49) and (1.70) for $\tilde{x} \in \partial D_N \cap \{\mathrm{Re}\tilde{x} \geq 0\}$:

$$\begin{aligned}
(1.82)\qquad & |\tilde{z}_2(\tilde{x}, \tilde{B})| \\
&\leq |1-\tilde{z}_0^2|^{1/2}\Big(2\sup_{|\tilde{x}|\leq N}|\Phi_2| + \frac{|\tilde{\Gamma}_2|}{2|\tilde{\Gamma}_0|}\Big)\int_1^{\tilde{z}_0(\tilde{x})} |1-\tilde{z}_0^2|^{-1/2}|d\tilde{z}_0| \\
&\leq 5(2N+3)\tilde{C}_0 N^{-p+1}\tilde{C}_2 \log N \\
&\leq 20\tilde{C}_0\tilde{C}_2 N^{-p+2}\log N \\
&\leq C_0 N^{-p+4}(\varepsilon N)^{-2}2!(C_0)^{-1}5\tilde{C}_0\tilde{C}_2 \log N,
\end{aligned}$$

where the integration path is taken as a straight line segment that connects 1 and $\tilde{x}$; thus this choice of the integration path together with the assumption on $\tilde{x}$ enables us to dominate the multivalued integral in the following manner:

$$(1.83)\qquad \sup_{\substack{\pm\mathrm{Re}\tilde{x}\geq 0\\ |\tilde{x}|\leq N}} \int_{\pm 1}^{\tilde{z}_0(\tilde{x})} |1-\tilde{z}_0^2|^{-1/2}|d\tilde{z}_0| \leq \tilde{C}_2 \log N,$$

where $\tilde{C}_2$ is a positive constant that is independent of N. Using the second representation of (1.49), we find that $\tilde{z}_2$ satisfies the same estimates with

(1.82) for $\tilde{x} \in \partial D_N \cap \{-\mathrm{Re}\tilde{x} \geq 0\}$. Therefore (1.82) holds on D_N. Then, since $|N\zeta^N| \leq N$ for $|\zeta| \leq 1-\varepsilon$, by taking $C_1 \geq (C_0)^{-1} 5\tilde{C}_0\tilde{C}_2$, we immediately have (1.58) for $k = 1$. Further, from (1.37), (1.70), (1.81) and (1.82), we obtain the following estimates on D_N:

$$(1.84)\quad \begin{aligned} \left|\frac{\partial \tilde{z}_2}{\partial \tilde{x}}\right| &\leq \left|\frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\tilde{z}_0}{\tilde{z}_0^2 - 1}\tilde{z}_2\right| + \left|\frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\tilde{\Gamma}_2}{2\tilde{\Gamma}_0}\right| + |\Phi_2| \\ &\leq \frac{N+1/2}{(N-3/2)^2} 40\tilde{C}_0\tilde{C}_2 N^{-p+2}\log N + 16\tilde{C}_0 N^{-p+1} + \tilde{C}_0 N^{-p+1} \\ &\leq \tilde{C}_0(320\tilde{C}_2 \log N + 17)N^{-p+1} \\ &\leq C_0 N^{-p+3}(\varepsilon N)^{-2} 2!(2C_0)^{-1}\tilde{C}_0(320\tilde{C}_2 + 17)\log N. \end{aligned}$$

Therefore, by taking $C_1 \geq (2C_0)^{-1}\tilde{C}_0(320\tilde{C}_2+17)$, we obtain (1.59) for $k = 1$. Finally, from (1.40) and (1.82), we obtain the following estimates on D_N:

$$(1.85)\quad \begin{aligned} |y_2| &\leq \left|\frac{2\tilde{z}_0}{\tilde{x}^2 - 1}\right| |\tilde{z}_2| \\ &\leq \frac{N+1/2}{(N-1)^2} 40\tilde{C}_0\tilde{C}_2 N^{-p+2}\log N \\ &\leq C_0 N^{-p+3}(\varepsilon N)^{-2} 2!(C_0)^{-1}\tilde{C}_0 160\tilde{C}_2 \log N. \end{aligned}$$

Hence we obtain (1.60) for $k = 1$ with $C_1 \geq (C_0)^{-1}\tilde{C}_0 160\tilde{C}_2$. In conclusion, we obtain (1.57) $\sim$ (1.60) for $k = 1$. Here we remark that, from the discussion above,

(1.86) we can take $C_0 > 0$ arbitrarily small by taking C_1 sufficiently large.

Next we show (1.57) $\sim$ (1.60) for $k = n$ $(2 \leq n \leq N-1)$ under the assumption that these estimates hold for $1 \leq k \leq n-1$. We first confirm the following estimates:

$$(1.87)\quad \|\Phi_{2n}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)! C_1^n (\log N)^{n-1},$$

where $\tilde{C}_0$ is a positive constant that is independent of n, N and ε. Let us consider the first term of $\Phi_{2n}^{(1)}$. From [Part I, Lemma 1.2.2] and the induction hypothesis, we obtain the following estimates:

$$
\begin{aligned}
(1.88)\qquad & \left\| \frac{1}{2}\frac{d\tilde z_0}{d\tilde x}\sum_{\mu=2}^{n}\sum_{|\vec\kappa|_\mu=n}^{*}(-1)^\mu\,\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu}\right\|_{\{\varepsilon\}} \\
&\le \frac{1}{2}\left\|\frac{d\tilde z_0}{d\tilde x}\right\|_{\{\varepsilon\}}\sum_{\mu=2}^{n}\sum_{|\vec\kappa|_\mu=n}^{*}\left\|\frac{y_{2\kappa_1}}{y_0}\right\|_{\{\varepsilon\}}\cdots\left\|\frac{y_{2\kappa_\mu}}{y_0}\right\|_{\{\varepsilon\}} \\
&\le \sum_{\mu=2}^{n}\sum_{|\vec\kappa|_\mu=n}^{*}C_0^\mu N^{n-\mu N}(\varepsilon N)^{-2n}(2\kappa_1)!\cdots(2\kappa_\mu)!(C_1\log N)^n \\
&\le \sum_{\mu=2}^{n}(4C_0)^\mu(2n-\mu+1)!N^{n-\mu N}(\varepsilon N)^{-2n}(C_1\log N)^n \\
&\le N^n(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n\sum_{\mu=2}^{n}\frac{(4C_0N^{-N})^\mu}{(\mu-2)!} \\
&\le 16e^{4C_0}C_0^2N^{n-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.
\end{aligned}
$$

Here we note that the same reasoning as in (1.88) entails the following estimates for $1\le k\le n-1$:

$$
\begin{aligned}
(1.89)\qquad & \left\|\sum_{\mu=1}^{k}\sum_{|\vec\kappa|_\mu=k}^{*}(-1)^\mu\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu}\right\|_{\{\varepsilon\}} \\
&\le 4e^{4C_0}C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k.
\end{aligned}
$$

Next we consider the second term of $\Phi_{2n}^{(1)}$. Since at least two of k_j's are non-zero, the factor C_0N^{-N} appears at least twice in the estimation of the term. For example, the following part of the term with $k_2=k_3=0$ is one of the essential terms in the estimation:

$$
\begin{aligned}
(1.90)\qquad & \left\|\frac{1}{2}\sum_{\substack{k_1+k_4=n\\1\le k_1,k_4\le n-1}}\frac{d\tilde z_{2k_1}}{d\tilde x}\sum_{\mu=1}^{k_4}\sum_{|\vec\kappa|_\mu=k_4}^{*}(-1)^\mu\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu}\right\|_{\{\varepsilon\}} \\
&\le 8e^{4C_0}C_0^2N^{n-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.
\end{aligned}
$$

On the other hand, when at least three of $k_1,\cdots,k_4$ are non-zero, the factor C_0N^{-N} appears at least three times. Then, since $C_0N^{-N}\ll 1$, we obtain better estimates than (1.90) for these terms. Therefore the second term of $\Phi_{2n}^{(1)}$ satisfies the following estimates for some positive constant $\tilde{C}_0$:

$$\begin{aligned}(1.91)\qquad &\Bigg\| \frac{1}{2\tilde{\Gamma}_0} \sum_{\substack{k_1+\cdots+k_4=n\\ k_1,\cdots,k_4\le n-1}} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \frac{d\tilde{z}_{2k_2}}{d\tilde{x}} \tilde{\Gamma}_{2k_3} \\ &\times \sum_{\mu=\min\{1,k_4\}}^{k_4} \sum_{|\vec{\kappa}|_\mu=k_4}^{*} (-1)^\mu \frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu} \Bigg\|_{\{\varepsilon\}} \\ &\le \tilde{C}_0 C_0^2 N^{n-2N} (\varepsilon N)^{-2n} (2n-1)!(C_1 \log N)^n.\end{aligned}$$

Finally, since $|\tilde{z}_0^2 - 1| \ge (|\tilde{x}| - 3/2)^2 \ge N/8$ holds for $|\zeta| = 1-\varepsilon$ $(0 < \varepsilon \le (2N)^{-1}\log N)$ (cf. (1.68) and (1.79)), the estimates of the third term of $\Phi_{2n}^{(1)}$ follows from the maximum modulus principle and the induction hypothesis as follows:

$$\begin{aligned}(1.92)\qquad &\Bigg\| \frac{1}{2} \frac{1}{\tilde{z}_0^2 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \sum_{k_1+k_2=n}^{*} \tilde{z}_{2k_1} \tilde{z}_{2k_2} \Bigg\|_{\{\varepsilon\}} \\ &\le 8C_0^2 N^{n+1-2N} (\varepsilon N)^{-2n} (2n-1)!(C_1 \log N)^n.\end{aligned}$$

We thus obtain the following estimates of $\Phi_{2n}^{(1)}$ from (1.88), (1.91) and (1.92) for some positive constant $\tilde{C}_0$:

$$\begin{aligned}(1.93)\qquad \|\Phi_{2n}^{(1)}\|_{\{\varepsilon\}} &\le \tilde{C}_0 C_0^2 N^{n+1-2N} (\varepsilon N)^{-2n} (2n-1)!(C_1 \log N)^n \\ &\le \tilde{C}_0 C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n-1)! C_1^n (\log N)^{n-1}\end{aligned}$$

Now we consider the estimation of $\Phi_{2n}^{(2)}$. We first show the following

Lemma 1.2. *Let $d\tilde{z}_{2k}/d\tilde{x}$ satisfy (1.59) for $0 < \varepsilon \le (2N)^{-1}\log N$. Then the following inequalities hold:*

$$(1.94)\qquad \left\| \frac{d^2 \tilde{z}_{2k}}{d\tilde{x}^2} \right\|_{\{\varepsilon\}} \le e^2 (1-\varepsilon)^{-N} C_0 N^{k-1-N} (\varepsilon N)^{-2k-1} (2k+1)!(C_1 \log N)^k,$$

$$\begin{aligned}(1.95)\qquad \left\| \frac{d^3 \tilde{z}_{2k}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}} &\le e^2 (1-\varepsilon)^{-2N} C_0 N^{k-2-N} \\ &\times \left(1 + \frac{\log N}{2k+2}\right) (\varepsilon N)^{-2k-2} (2k+2)!(C_1 \log N)^k.\end{aligned}$$

Proof. Appealing to the maximum modulus principle, it is enough to show (1.94) and (1.95) for $|\zeta| = 1-\varepsilon$. We first note the following relation:

$$\frac{d^2\tilde{z}_{2k}}{d\tilde{x}^2} = \frac{\zeta^{-N+1}}{N^2}\frac{d}{d\zeta}\frac{d\tilde{z}_{2k}}{d\tilde{x}}, \tag{1.96}$$

$$\frac{d^3\tilde{z}_{2k}}{d\tilde{x}^3} = \left(\frac{\zeta^{-2N+2}}{N^4}\frac{d^2}{d\zeta^2} + \frac{-N+1}{N^4}\zeta^{-2N+1}\frac{d}{d\zeta}\right)\frac{d\tilde{z}_{2k}}{d\tilde{x}}. \tag{1.97}$$

We use the following representation:

$$\frac{d^j}{d\zeta^j}\frac{d\tilde{z}_{2k}}{d\tilde{x}} = \frac{j!}{2\pi\sqrt{-1}}\int_{|\tilde{\zeta}-\zeta|=(k+1)^{-1}\varepsilon}\frac{d\tilde{z}_{2k}}{d\tilde{x}}\frac{d\tilde{\zeta}}{(\tilde{\zeta}-\zeta)^{j+1}}. \tag{1.98}$$

We immediately find that the integral path of (1.98) is contained in $|\tilde{\zeta}| \leq 1-\tilde{\varepsilon}$ with $\tilde{\varepsilon} = k\varepsilon/(k+1)$. Since

$$\begin{aligned}\left\|\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\tilde{\varepsilon}\}} &\leq C_0N^{k-N}(\tilde{\varepsilon}N)^{-2k}(2k)!(C_1\log N)^k \\ &= C_0N^{k-N}\left(1+\frac{1}{k}\right)^{2k}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k \\ &\leq e^2C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k\end{aligned} \tag{1.99}$$

follows from (1.59), we obtain the following estimates from (1.98):

$$\left\|\frac{d^j}{d\zeta^j}\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\varepsilon\}} \leq j!(k+1)^j\varepsilon^{-j}e^2C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k. \tag{1.100}$$

Then, from (1.100), we obtain the estimates of (1.96) and (1.97) as follows:

$$\begin{aligned}&\left\|\frac{\zeta^{-N+1}}{N^2}\frac{d}{d\zeta}\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\varepsilon\}} \\ &\leq e^2(1-\varepsilon)^{-N}C_0N^{k-1-N}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k,\end{aligned} \tag{1.101}$$

$$\left\|\left(\frac{\zeta^{-2N+2}}{N^4}\frac{d^2}{d\zeta^2} + \frac{-N+1}{N^4}\zeta^{-2N+1}\frac{d}{d\zeta}\right)\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\varepsilon\}} \tag{1.102}$$

$$\leq e^2(1-\varepsilon)^{-2N}C_0N^{k-2-N}(\varepsilon N)^{-2k-2}(2k+2)!(C_1\log N)^k$$
$$+2e^2(1-\varepsilon)^{-2N}C_0N^{k-2-N}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k.$$

Since $\varepsilon N \leq 2^{-1}\log N$, (1.94) and (1.95) immediately follow from (1.101) and (1.102).

□

We return to the estimation of $\Phi^{(2)}_{2n}$. Let us consider the first term of $\Phi^{(2)}_{2n}$. By the same reasoning as the estimation of (1.89), the following holds for $k \geq 1$:

$$\Big\|\sum_{\mu=1}^{k}\sum_{|\vec{\kappa}|_\mu=k}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\|_{\{\varepsilon\}} \leq 8e^{8C_0}C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k. \tag{1.103}$$

By the discussion similar to the estimation of (1.91), we find that the terms with $k_1=0$ or $k_2=0$ are essential in the estimation. In particular, since (1.73) holds, we see that the following term with $k_2=0$ is the worst contribution:

$$\begin{aligned}\Big\|\frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\frac{d^3\tilde{z}_{2(n-1)}}{d\tilde{x}^3}\Big\|_{\{\varepsilon\}} &\leq 2(N^2(1-\varepsilon)^{2N}+3)e^2(1-\varepsilon)^{-2N}C_0N^{n-3-N}\\ &\quad\times\left(1+\frac{\log N}{2n}\right)(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\ &\leq e^2C_0N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}.\end{aligned} \tag{1.104}$$

Therefore, having (1.73) in mind, we obtain the following estimates for some positive constant $\tilde{C}_0$:

$$\begin{aligned}\Big\|\frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\sum_{k_1+k_2=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\frac{d^3\tilde{z}_{2k_1}}{d\tilde{x}^3}\\ \times\sum_{\mu=\min\{1,k_2\}}^{k_2}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\|_{\{\varepsilon\}}\end{aligned} \tag{1.105}$$

$$\leq \tilde{C}_0 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

By the same reasoning, we find that the following estimates for the second term of $\Phi_{2n}^{(2)}$ follows from (1.94) and [Part I, Lemma 1.2.2]:

$$\begin{aligned}
(1.106) \qquad & \Big\| \frac{3(\tilde{z}_0^2 - 1)}{8\tilde{\Gamma}_0} \sum_{k_1+k_2+k_3=n-1} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3} \frac{d^2 \tilde{z}_{2k_1}}{d\tilde{x}^2} \frac{d^2 \tilde{z}_{2k_2}}{d\tilde{x}^2} \\
& \times \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu = k_3}^{*} (-1)^\mu (\mu+1) \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \Big\|_{\{\varepsilon\}} \\
& \leq \tilde{C}_0 C_0 N^{n-1-2N} (\varepsilon N)^{-2n} (2n-1)! (C_1 \log N)^{n-1}.
\end{aligned}$$

Thus we see that the following estimates hold for some positive constant $\tilde{C}_0$:

$$(1.107) \qquad \|\Phi_{2n}^{(2)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

Finally we consider the estimation of $\Phi_{2n}^{(3)}$. Let us consider the first term of $\Phi_{2n}^{(3)}$. Since it is holomorphic on $|\zeta| < 1$, it suffices to estimate it for $|\zeta| = 1 - \varepsilon$. We first note that, since $|\tilde{z}_0 - 1| \geq 4^{-1}\sqrt{N}$ holds on $|\zeta| = 1 - \varepsilon$, we find the following estimates:

$$\begin{aligned}
(1.108) \qquad & \Big\| \sum_{\mu=1}^{k} \sum_{|\vec{\kappa}|_\mu = k}^{*} (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0 - 1)^\mu} \Big\|_{\{\varepsilon\}} \\
& \leq 32 e^{32 C_0} C_0 N^{k-N+1/2} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k.
\end{aligned}$$

By the same discussion as the estimation of (1.91), we find that the terms with one of k_j's being $n-1$ are essential in the estimation. In particular, since $32 e^{32C_0} N^{-1/2} \ll 1$, comparison of (1.108) and (1.59) entails that the worst is the term with $k_3 = n-1$, which can be estimated as follows:

$$\begin{aligned}
(1.109) \qquad & \Big\| \frac{\tilde{z}_0 + 1}{\tilde{z}_0 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \frac{g_+(a)}{2\tilde{\Gamma}_0} \sum_{\mu=1}^{n-1} \sum_{|\vec{\kappa}|_\mu = n-1}^{*} (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0 - 1)^\mu} \Big\|_{\{\varepsilon\}} \\
& \leq \frac{N(1-\varepsilon)^N + 3/2}{N(1-\varepsilon)^N - 3/2} 4^3 \tilde{C}_1 e^{32 C_0} C_0 \\
& \quad \times N^{n-N-1/2} (\varepsilon N)^{-2(n-1)} (2n-2)! (C_1 \log N)^{n-1}
\end{aligned}$$

$$\le 4^3\tilde{C}_1 e^{32C_0} C_0 N^{n-N-1/2}\frac{(\log N)^2}{n^2}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}$$
$$\le 4^5\tilde{C}_1 e^{32C_0} C_0 N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1},$$

where $\tilde{C}_1$ is a positive constant that satisfies (1.75). In this way, we can obtain the estimates of the first term of $\Phi_{2n}^{(3)}$. On the other hand, we immediately find that the second term also satisfies the same estimates with the first term. Therefore we find that the following estimates hold for $\Phi_{2n}^{(3)}$ with a positive constant $\tilde{C}_0$:

$$\|\Phi_{2n}^{(3)}\|_{\{\varepsilon\}} \le \tilde{C}_0 C_0 N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}. \tag{1.110}$$

By taking $C_1^{-1} \le C_0$ and summing up (1.93), (1.107) and (1.110), we obtain (1.87).

Now we confirm (1.57) $\sim$ (1.60) for $k = n$. We first note that the following estimates follow from (1.48) and (1.87):

$$\begin{aligned}|\tilde{\Gamma}_{2n}(\tilde{B})| &\le \frac{4|\tilde{\Gamma}_0|}{\pi}\int_1^{-1}|1-\tilde{z}_0^2|^{-1/2}|d\tilde{z}_0|\|\Phi_{2n}\|_{\{\varepsilon\}} \\ &\le 8\tilde{C}_0 C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)!C_1^n(\log N)^{n-1}.\end{aligned} \tag{1.111}$$

Then, by taking C_0 sufficiently small so that they satisfy $8\tilde{C}_0 C_0 < 1$, we obtain (1.57). Next, from (1.49), (1.87) and (1.111), we obtain the following estimates on $\{|\tilde{x}| = N(1-\varepsilon)^N\} \cap \{\mathrm{Re}\tilde{x} \ge 0\}$:

$$\begin{aligned}|\tilde{z}_{2n}(\tilde{x},\tilde{B})| &\le |1-\tilde{z}_0^2|^{1/2}\int_1^{\tilde{x}}|1-\tilde{z}_0^2|^{-1/2}|d\tilde{z}_0|\Big(2\|\Phi_{2n}\|_{\{\varepsilon\}} + \frac{|\tilde{\Gamma}_{2n}|}{2|\tilde{\Gamma}_0|}\Big) \\ &\le 20(1-\varepsilon)^N\tilde{C}_0\tilde{C}_2 C_0^2 N^{n+1-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^n,\end{aligned} \tag{1.112}$$

where $\tilde{C}_2$ is a positive constant appearing in (1.83). By the same discussion, we find from the second representation of (1.49) that (1.112) also holds on $\{|\tilde{x}| = N(1-\varepsilon)^N\} \cap \{-\mathrm{Re}\tilde{x} \ge 0\}$. Since $\tilde{z}_{2n}$ is holomorphic on $\{|\tilde{x}| \le N(1-\varepsilon)^N\}$, we see that (1.112) holds there. Hence, by taking C_0 so that $20\tilde{C}_0\tilde{C}_2 C_0 < 1$ holds, we obtain (1.58) for $k = n$. Then, using the relation (1.37), we obtain the following estimates from (1.87), (1.111) and (1.112):

$$\Big\|\frac{\partial\tilde{z}_{2n}}{\partial\tilde{x}}\Big\|_{\{\varepsilon\}} \le 2\frac{N(1-\varepsilon)^N + 1/2}{(N(1-\varepsilon)^N - 3/2)^2}\|\tilde{z}_{2n}\|_{\{\varepsilon\}} + 2|\tilde{\Gamma}_{2n}| + \|\Phi_{2n}\|_{\{\varepsilon\}} \tag{1.113}$$

$$\leq (320\tilde{C}_0\tilde{C}_2 + 9\tilde{C}_0)C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1 \log N)^n.$$

Therefore, by taking C_0 so that $(320\tilde{C}_0\tilde{C}_2 + 9\tilde{C}_0)C_0 < 1$ holds, we find that $\tilde{z}_{2n}$ satisfies (1.59). Furthermore, by the maximum modulus principle, we obtain the following estimates from (1.40), (1.58), (1.69) and (1.112):

$$\begin{aligned}(1.114)\quad & \|y_{2n}\|_{\{\varepsilon\}} \\ & \leq \frac{1}{N^2(1-\varepsilon)^{2N}-1}\Big(2\|\tilde{z}_0\|_{\{\varepsilon\}}\|\tilde{z}_{2n}\|_{\{\varepsilon\}} + \sum_{k=1}^{n-1}\|\tilde{z}_{2k}\|_{\{\varepsilon\}}\|\tilde{z}_{2(n-k)}\|_{\{\varepsilon\}}\Big) \\ & \leq \frac{4}{N^2(1-\varepsilon)^{2N}}\Big(80N(1-\varepsilon)^{2N}\tilde{C}_0\tilde{C}_2 + 2n^{-1}N^{1-N}\Big) \\ & \quad \times C_0^2 N^{n+1-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^n \\ & \leq 4(80\tilde{C}_0\tilde{C}_2 + 1)C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^n.\end{aligned}$$

Therefore, by taking C_0 so that it satisfies $4(80\tilde{C}_0\tilde{C}_2 + 1)C_0 < 1$, we obtain (1.60) for $k = n$. Thus the induction proceeds and we obtain (1.57) $\sim$ (1.60) for $1 \leq k \leq N-1$.

Now, we confirm (1.61) $\sim$ (1.64) for $k \geq N$. We first remark that, from (1.57) $\sim$ (1.60), we find that (1.61) $\sim$ (1.64) also hold for $1 \leq k \leq N-1$. Hence we show (1.61) $\sim$ (1.64) for $k = n$ $(n \geq N)$ under the assumption that these estimates hold for $1 \leq k \leq n-1$. By the same discussion with the derivation of (1.57) $\sim$ (1.60) from (1.87), it suffices to show the following estimates:

$$(1.115)\qquad \|\Phi_{2n}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{-1}(\varepsilon N)^{-2n}(2n)!C_1^n(\log N)^{n-1},$$

where $\tilde{C}_0$ is some positive constant. We first confirm the following estimates:

$$(1.116)\qquad \|\Phi_{2n}^{(1)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{-1}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.$$

Then, since $\log N \leq n$, we find that $\Phi_{2n}^{(1)}$ satisfies (1.115). As in the derivation of (1.93), the following term is essential in the estimation of $\Phi_{2n}^{(1)}$:

$$\begin{aligned}(1.117)\qquad & \Big\|\frac{1}{2}\frac{1}{\tilde{z}_0^2-1}\frac{d\tilde{z}_0}{d\tilde{x}}\sum_{k_1+k_2=n}^{*}\tilde{z}_{2k_1}\tilde{z}_{2k_2}\Big\|_{\{\varepsilon\}} \\ & \leq 8N^{-1}C_0^2(\varepsilon N)^{-2n}(C_1\log N)^n \sum_{k_1+k_2=n}^{*}(2k_1)!(2k_2)!\end{aligned}$$

$$\le 32C_0^2N^{-1}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.$$

Here we used the fact that $|\tilde{z}_0^2-1|\ge N/8$ holds for $|\zeta|=1-\varepsilon$. In this way, we can show that the first and the second term of $\Phi_{2n}^{(1)}$ also satisfy (1.116) by the same discussion with the estimation of (1.88) and (1.91).

Next, we show the following estimates:

$$\|\Phi_{2n}^{(2)}\|_{\{\varepsilon\}} \le \tilde{C}_0C_0N^{-1}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}. \tag{1.118}$$

We first note that, by the same discussion with the proof of Lemma 1.2, we obtain the following estimates for $k=1,2,\cdots$ from (1.63):

$$\left\|\frac{d^2\tilde{z}_{2k}}{d\tilde{x}^2}\right\|_{\{\varepsilon\}} \le e^2(1-\varepsilon)^{-N}C_0N^{-2}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k, \tag{1.119}$$

$$\begin{aligned}\left\|\frac{d^3\tilde{z}_{2k}}{d\tilde{x}^3}\right\|_{\{\varepsilon\}} &\le e^2(1-\varepsilon)^{-2N}C_0N^{-3}\\ &\quad\times\left(1+\frac{\log N}{2k+2}\right)(\varepsilon N)^{-2k-2}(2k+2)!(C_1\log N)^k.\end{aligned} \tag{1.120}$$

Let us consider the first term of $\Phi_{2n}^{(2)}$, which is essential in the estimation of $\Phi_{2n}^{(2)}$. Since $\log N\le n$, we find the following estimates from (1.120):

$$\left\|\frac{d^3\tilde{z}_{2(n-1)}}{d\tilde{x}^3}\right\|_{\{\varepsilon\}} \le \frac{3e^2}{2}(1-\varepsilon)^{-2N}C_0N^{-3}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}. \tag{1.121}$$

And, since $\log N\le N$, we find the following estimates from (1.120) for $1\le k\le n-2$:

$$\left\|\frac{d^3\tilde{z}_{2(k-1)}}{d\tilde{x}^3}\right\|_{\{\varepsilon\}} \le e^2(1-\varepsilon)^{-2N}C_0N^{-2}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^{k-1}. \tag{1.122}$$

Then, by the same reasoning with the estimates of (1.105), we obtain the following estimates for the first term of $\Phi_{2n}^{(2)}$:

$$\left\|\frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\left\{\frac{d^3\tilde{z}_{2(n-1)}}{d\tilde{x}^3}\right.\right. \tag{1.123}$$

$$
\begin{aligned}
&+\sum_{k_1+k_2=n-1}^{*}\frac{d^3\tilde{z}_{2k_1}}{d\tilde{x}^3}\sum_{\mu=\min\{1,k_2\}}^{k_2}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\\
&+\frac{d^3\tilde{z}_0}{d\tilde{x}^3}\sum_{\mu=1}^{n-1}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\}\Big\|_{\{\varepsilon\}}\\
&\le 8N^2(1-\varepsilon)^{2N}\{e^2(1-\varepsilon)^{-2N}C_0N^{-3}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\
&\quad+8e^{8C_0+2}(1-\varepsilon)^{-2N}C_0^2N^{-3}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\
&\quad+8e^{8C_0}C_0N^{-1-p}(\varepsilon N)^{-2(n-1)}(2n-2)!(C_1\log N)^{n-1}\}\\
&\le 8C_0N^{-1}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\
&\qquad\times\{e^2+8e^{8C_0+2}C_0+e^{8C_0}N^{-p+2}n^{-2}(\log N)^2\}.
\end{aligned}
$$

Since $p \ge N+2$, we immediately find that the first term of $\Phi_{2n}^{(2)}$ satisfies (1.118). In the same way, from (1.119), we can show the following estimates:

$$
\begin{aligned}
(1.124)\qquad &\Big\|\frac{3(\tilde{z}_0^2-1)}{8\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3}\frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2}\frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2}\\
&\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\|_{\{\varepsilon\}}\\
&\le \tilde{C}_0C_0N^{-2}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^{n-1}.
\end{aligned}
$$

Hence, from (1.123) and (1.124), we obtain (1.118).

Finally, by the same discussion with the estimation of (1.109), we obtain the following estimates:

$$
(1.125)\qquad \|\Phi_{2n}^{(3)}\|_{\{\varepsilon\}} \le \tilde{C}_0C_0N^{-3/2}(\varepsilon N)^{-2n+2}(2n-2)!(C_1\log N)^{n-1}.
$$

Then, since $N^{1/2} \le n$ and $(\varepsilon N)^2 \le n$, we find that $\Phi_{2n}^{(3)}$ satisfies (1.115).

Summing up, we have confirmed (1.61) $\sim$ (1.64) for $k = n$. Thus the induction proceeds. This completes the proof of Lemma 1.1, completing the proof of Theorem 1.1.

□

As is shown in [KT], we can deduce the following Theorem 1.2 from Theorem 1.1:

Theorem 1.2. *Let $\tilde{S}$ and S respectively be a solution of*

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial x} = \eta^2 \left(\frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \tag{1.126}$$

and

$$S^2 + \frac{\partial S}{\partial z} = \eta^2 \left(\frac{a\Gamma}{z^2 - a^2} + \eta^{-2} \left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2} \right) \right), \tag{1.127}$$

and suppose that

$$\arg \tilde{S}_{-1}(x, a, A, B) = \arg \left(\frac{\partial z_0}{\partial x} S_{-1}(z_0(x, a, A, B), a, \Gamma_0(a, A, B)) \right) \tag{1.128}$$

holds. Then they satisfy

$$\begin{aligned} &\tilde{S}_{\mathrm{odd}}(x, a, A, B, \eta) \\ &= \left(\frac{\partial z}{\partial x} \right) S_{\mathrm{odd}}(z(x, a, A, B, \eta), a, \Gamma(a, A, B, \eta), \eta) \end{aligned} \tag{1.129}$$

on $E^2_{r_1, r_2}$, where $\tilde{S}_{\mathrm{odd}}$ and S_{odd} respectively denote the odd part of $\tilde{S}$ and S.

We also have the following

Theorem 1.3. *Let $\tilde{\psi}_\pm(x, a, A, B, \eta)$ be WKB solutions of the generic (i.e., $a \neq 0$) Mathieu equation (1.1) that are normalized at a simple pole $x = a$ as*

$$\tilde{\psi}_\pm(x, a, A, B, \eta) = \frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}} \exp \left(\pm \int_a^x \tilde{S}_{\mathrm{odd}} dx \right), \tag{1.130}$$

and $\psi_\pm(z, a, \Gamma, \eta)$ be WKB solutions of the Legendre equation (1.4) that is normalized at a simple pole $z = a$ as

$$\psi_\pm(z, a, \Gamma, \eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp \left(\pm \int_a^z S_{\mathrm{odd}} dz \right). \tag{1.131}$$

Then they satisfy the following relation (1.132) *on an open set* $E^2_{r_1,r_2}$ *given by* (1.8)*:*

$$\tilde{\psi}_{\pm}(x,a,A,B,\eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2}\psi_{\pm}(z(x,a,A,B,\eta),a,\Gamma(a,A,B,\eta),\eta), \tag{1.132}$$

where $z(x,a,A,B,\eta)$ *and* $\Gamma(a,A,B,\eta)$ *are the series constructed in Theorem 1.1.*

We have so far discussed how WKB solutions of (1.1) are related to WKB solutions of (1.4). But we need in Section 2 the Legendre equation in the form (1.2). Here we discuss how WKB solutions of (1.4) and those of (1.2) are related; as we will see below the relation can be found in a straightforward manner. For the sake of simplicity of description we consider the situation when the parameter Γ in (1.4) is a genuine constant; this restriction does not cause any problems in our later discussion, as appropriate use of microdifferential operators will enable us to relate (1.4) with Γ being a genuine constant and (1.4) with Γ being infinite series. (See Proposition 3.3.) To relate (1.4) and (1.2) we define an infinite series

$$\Lambda(a,\Gamma,\eta) = \sum_{n=0}^{\infty}\Lambda_n(a,\Gamma)\eta^{-n} \tag{1.133}$$

and functions $\mu(a)$ and $\nu(a)$ of a by

$$\Lambda = \sqrt{\Gamma + (\sqrt{a}\eta)^{-2}\left(g_+(a)+g_-(-a)+\frac{1}{4}\right)} - \frac{(\sqrt{a}\eta)^{-1}}{2}, \tag{1.134}$$

$$\mu = \sqrt{1+2(g_+(a)+g_-(-a))}, \tag{1.135}$$

$$\nu = 2(g_+(a)-g_-(-a)). \tag{1.136}$$

Since $\Lambda(a,\Gamma,\eta)$ satisfy

$$a\Gamma = a\Lambda^2 + \eta^{-1}\sqrt{a}\Lambda - \eta^{-2}(g_+(a)+g_-(-a)), \tag{1.137}$$

we immediately obtain (1.4) from (1.2) by choosing Λ, μ and ν in (1.2) respectively by (1.134),(1.135) and (1.136). Therefore we find the following

Proposition 1.1. *Let* $T_{\mathrm{odd}}(z, a, \Lambda, \mu, \nu, \eta)$ *and* $\phi_\pm(z, a, \Lambda, \mu, \nu, \eta)$ *respectively be the odd part of the solution of the Riccati equation*

$$T^2 + \frac{\partial T}{\partial z} = \eta^2 \left(\frac{a\Lambda^2}{z^2 - a^2} + \eta^{-1} \frac{\sqrt{a}\Lambda}{z^2 - a^2} + \eta^{-2} \frac{az\nu + a^2(\mu^2 - 1)}{(z^2 - a^2)^2} \right) \tag{1.138}$$

and WKB solutions of (1.2) *that are normalized at a simple pole* $z = a$ *as*

$$\phi_\pm(z, a, \Lambda, \mu, \nu, \eta) = \frac{1}{\sqrt{T_{\mathrm{odd}}}} \exp\left(\pm \int_a^z T_{\mathrm{odd}} dz \right). \tag{1.139}$$

Then the following relations hold:

$$S_{\mathrm{odd}}(z, a, \Gamma, \eta) = T_{\mathrm{odd}}(z, a, \Lambda(a, \Gamma, \eta), \mu(a), \nu(a), \eta), \tag{1.140}$$

$$\psi_\pm(z, a, \Gamma, \eta) = \phi_\pm(z, a, \Lambda(a, \Gamma, \eta), \mu(a), \nu(a), \eta), \tag{1.141}$$

where the infinite series $\Lambda(a, \Gamma, \eta)$ *and the functions* $\mu(a)$ *and* $\nu(a)$ *are those given by* (1.134), (1.135) *and* (1.136) *respectively.*

Remark 1.5. Since $\Lambda(a, \Gamma, \eta)$ given by (1.134) is a convergent power series in η, $\Lambda_n(a, \Gamma)$ $(n \geq 1)$ satisfy the following estimates: There exists a positive constant C such that

$$|\Lambda_n(a, \Gamma)| \leq \sqrt{|\Gamma|} \left(\frac{C}{\sqrt{|a\Gamma|}} \right)^n \tag{1.142}$$

holds for $a\Gamma \neq 0$ and $n \geq 1$.

2. Analytic properties of Borel transformed WKB solutions of the Legendre equation with a large parameter

The main purpose of this section is to present analytic properties of Borel transformed WKB solutions of (1.2) with genuine constants a, Λ, μ and ν. To begin with, we show the following important

Proposition 2.1. *Let* $T_{\mathrm{odd}}(z, a, \Lambda, \eta)$ *be the odd part of the solution of* (1.138) *whose top degree part* $T_{-1}(z, a, \Lambda)$ *is chosen so that it is positive for positive* a, $z(> a)$ *and* Λ. *Then we have*

$$\oint_\gamma T_{\mathrm{odd}}(z, a, \Lambda, \eta) dz = 2\pi i \sqrt{a}\Lambda\eta + \pi i, \tag{2.1}$$

where γ *is a closed curve that encircles two simple poles* $z = \pm a$ *counter-clockwise.*

Proof. Let

$$T^{(\pm)}(z,a,\Lambda,\eta)=\sum_{n=-1}^{\infty}T_n^{(\pm)}(z,a,\Lambda)\eta^{-n} \tag{2.2}$$

be the solutions of (1.138) whose top degree parts $T_{-1}^{(\pm)}(z,a,\Lambda)$ are respectively given by

$$T_{-1}^{(\pm)}(z,a,\Lambda)=\pm\sqrt{\frac{a\Lambda^2}{z^2-a^2}}. \tag{2.3}$$

Then $T_0^{(\pm)}$ and $T_1^{(\pm)}$ are respectively given by

$$T_0^{(\pm)}=\frac{1}{2}\frac{z}{z^2-a^2}\pm\frac{1}{2}\frac{1}{\sqrt{z^2-a^2}} \tag{2.4}$$

and

$$T_1^{(\pm)}=\pm\frac{4a\nu z+a^2(4\mu^2-1)}{8\sqrt{a}\Lambda(z^2-a^2)^{3/2}}. \tag{2.5}$$

Further we can inductively confirm that $T_n^{(\pm)}$ $(n\geq 2)$ have the following form:

$$\begin{aligned}T_n^{(\pm)}=&\sum_{2\leq p\leq n+2}c_{p,n}^{(\pm)}(z^2-a^2)^{-p/2}\\&+\sum_{3\leq p\leq n+2}d_{p,n}^{(\pm)}z(z^2-a^2)^{-p/2},\end{aligned} \tag{2.6}$$

where $c_{p,n}^{(\pm)}$ and $d_{p,n}^{(\pm)}$ are constants. Hence, by noting that

$$\oint_\gamma\frac{dz}{\sqrt{z^2-a^2}}=2\pi i \tag{2.7}$$

and

$$\oint_\gamma T_n^{(\pm)}dz=0 \tag{2.8}$$

hold for $n\geq 1$, we immediately obtain (2.1).

□

Now we consider the Voros coefficient

$$V(a,\Lambda,\eta)=\sum_{n=1}^{\infty}V_n\eta^{-n} \tag{2.9}$$

of (1.2), which is, by definition, given by

$$\int_a^{\infty}\left(T_{\mathrm{odd}}-\eta T_{-1}-\frac{1}{2z}\right)dz \tag{2.10}$$

(cf. [DP], [AKT]). Let $\phi_{\pm}^{(\infty)}$ be WKB solutions of (1.2) that are normalized at infinity as

$$\phi_{\pm}^{(\infty)}=\frac{z^{\pm 1/2}}{\sqrt{T_{\mathrm{odd}}}}e^{\pm\eta y_+}\exp\left[\pm\int_{\infty}^{z}\left(T_{\mathrm{odd}}-\eta T_{-1}-\frac{1}{2z}\right)dz\right], \tag{2.11}$$

where

$$y_+(z,a,\Lambda)=\int_a^z\sqrt{\frac{a\Lambda^2}{z^2-a^2}}dz. \tag{2.12}$$

Then WKB solutions (1.139) of (1.2) that are normalized at $z=a$ as (1.139) are written by V and $\phi_{\pm}^{(\infty)}$ as follows:

$$\phi_{\pm}=a^{\mp 1/2}\exp\left(\pm V\right)\phi_{\pm}^{(\infty)}. \tag{2.13}$$

An important property of $\phi_{\pm}^{(\infty)}$ is that they are Borel summable when

(2.14) the path of integration of (2.11) from ∞ to z can be deformed so that it does not intersect Stokes curves of (1.2).

See [KoS] for the proof of the Borel summability of $\phi_{\pm}^{(\infty)}$. Hence the representation (2.13) of $\phi_{\pm}$ entails that the calculation of the alien derivative of $\phi_{\pm}$ is reduced to that of V. Fortunately the explicit form of V has been given by T. Koike ([Ko]) as follows:

(2.15)
$$V_n=\frac{1}{n(n+1)(\sqrt{a}\Lambda)^n}\left[B_{n+1}+\sum_{\substack{k+2l=n+1\\k,l\geq 0}}\frac{(n+1)!}{k!(2l)!}B_k\left\{\left(\frac{1}{2}\right)^{2l}-\theta_+^{2l}-\theta_-^{2l}\right\}\right]$$

for $n \geq 1$, where B_n $(n = 0, 1, 2, \cdots)$ are Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \tag{2.16}$$

and

$$\theta_{\pm}(\mu, \nu) = \sqrt{\frac{\mu^2 \pm \sqrt{\mu^4 - \nu^2}}{2}}. \tag{2.17}$$

In [Ko] the derivation of (2.15) is done in a parallel way to the computation of the Voros coefficient of the Weber equation and the Whittaker equation. See [SS] and [T] (resp., [KoT]) for the computation of the Voros coefficient of the Weber equation (resp., the Whittaker equation). Hence the Borel transform $V_B(a, \Lambda, y)$ of V is concretely given by

$$\begin{aligned} V_B = & \frac{1}{y(\exp(y/\sqrt{a}\Lambda) - 1)} \\ & \times \left\{ 1 + \cosh\left(\frac{y}{2\sqrt{a}\Lambda}\right) - \cosh\left(\frac{\theta_+ y}{\sqrt{a}\Lambda}\right) - \cosh\left(\frac{\theta_- y}{\sqrt{a}\Lambda}\right) \right\}. \end{aligned} \tag{2.18}$$

It immediately follows from (2.18) that V_B behaves as

$$V_B = \frac{1}{2\sqrt{a}\Lambda}\left(\frac{1}{4} - (\theta_+^2 + \theta_-^2)\right) + O(y) \tag{2.19}$$

near $y = 0$ and

$$V_B = \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{2m\pi i(y - 2m\pi i\sqrt{a}\Lambda)} + O(1) \tag{2.20}$$

near $y = 2m\pi i\sqrt{a}\Lambda$ for $m \in \mathbb{Z}\setminus\{0\}$. Therefore V_B is singular at $y = 2m\pi i\sqrt{a}\Lambda$ $(m \in \mathbb{Z}\setminus\{0\})$ and it has simple poles there.

Now let us compute the alien derivative

$$\Delta V = \sum_{m \geq 1} \Delta_{y = 2m\pi i\sqrt{a}\Lambda} V \tag{2.21}$$

of the Voros coefficient V by using the alien calculus initiated by [Ec] and developed by [P], [DP] and [Sa]. Since V_B is single-valued and only has simple pole singularities, $\Delta_{y=2m\pi i\sqrt{a}\Lambda}V$ is given by the residue of V_B at $y = 2m\pi i\sqrt{a}\Lambda$, i.e.,

$$\Delta_{y=2m\pi i\sqrt{a}\Lambda}V = \frac{1+(-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m}. \tag{2.22}$$

Then, by employing the alien calculus, we find

$$\begin{aligned}\Delta_{y=2m\pi i\sqrt{a}\Lambda}&\exp(\pm V)\\ &= \pm\frac{1+(-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m}\exp(\pm V).\end{aligned} \tag{2.23}$$

Noting the fact that

$$\Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(e^{\mp\eta y_+}a^{\mp 1/2}\phi_\pm^{(\infty)}\right) = 0 \tag{2.24}$$

hold for $m \geq 1$ under the condition (2.14), we find (2.13) entails the following relations when (2.14) is satisfied:

$$\begin{aligned}\Delta_{y=2m\pi i\sqrt{a}\Lambda}&\left(e^{\mp\eta y_+}\phi_\pm\right)\\ &= \Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(e^{\mp\eta y_+}a^{\mp 1/2}\exp(\pm V)\phi_\pm^{(\infty)}\right)\\ &= e^{\mp\eta y_+}a^{\mp 1/2}\phi_\pm^{(\infty)}\Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(\exp(\pm V)\right)\\ &= \pm\frac{1+(-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m}\\ &\quad\times e^{\mp\eta y_+}a^{\mp 1/2}\phi_\pm^{(\infty)}\exp(\pm V)\\ &= \pm\frac{1+(-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m}e^{\mp\eta y_+}\phi_\pm.\end{aligned} \tag{2.25}$$

Summing up all these, we obtain the following

Theorem 2.1. *Let $\phi_\pm(z,\eta)$ denote the WKB solutions of the Legendre equation* (1.2) *that are normalized at a simple pole $z = a$ as in* (1.139). *Then their Borel transform $\phi_{\pm,B}(z,y)$ are singular at*

$$y = \mp y_+(z) + 2m\pi i\sqrt{a}\Lambda \quad (m = 0, \pm 1, \pm 2, \cdots), \tag{2.26}$$

where $y_+(z)$ *is the function given by* (2.12)*, and its alien derivative there satisfies the following relation* (2.27) *for* z *that can be connected with* $z = \infty$ *by a path that is contained in the interior of a Stokes region of the Legendre equation.*

$$(2.27)\qquad \left(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a}\Lambda}\phi_\pm\right)_B(z,y) = \pm\, \Xi_m(\mu,\nu)\phi_{\pm,B}(z, y - 2m\pi i\sqrt{a}\Lambda),$$

where

$$(2.28)\qquad \Xi_m(\mu,\nu) = \frac{1}{m}\left\{1 + (-1)^m - \cosh\left(2\pi i m\sqrt{\frac{\mu^2+\sqrt{\mu^4-\nu^2}}{2}}\right)\right.$$
$$\left. - \cosh\left(2\pi i m\sqrt{\frac{\mu^2-\sqrt{\mu^4-\nu^2}}{2}}\right)\right\}.$$

3. Analytic properties of Borel transformed WKB solutions of the Mathieu equation — properties relevant to simple poles

The principal aim of this section is to deduce analytic properties of Borel transformed WKB solutions of the Mathieu equation (1.1) for $a \neq 0$ and $A \neq 0$ that are relevant to its two simple poles from those of the Legendre equation (1.2) through the transformation obtained in Section 1. To begin with, we show a result corresponding to Proposition 2.1 for the Mathieu equation. First, combining Proposition 1.1 and Proposition 2.1 we immediately find

$$(3.1)\qquad \oint_\gamma S_{\mathrm{odd}}(z,a,\Gamma,\eta)dz = 2\pi i\sqrt{a}\Lambda(a,\Gamma,\eta)\eta + \pi i,$$

where γ is the path given in Proposition 2.1. Therefore Proposition 3.1 below follows from Theorem 1.2.

Proposition 3.1. *Let* $\tilde{S}_{\mathrm{odd}}(x,a,A,B,\eta)$ *be the odd part of the solution of* (1.126) *whose top degree part* $\tilde{S}_{-1}(x,a,A,B)$ *is chosen so that it satisfies* (1.128)*. Then we have*

$$(3.2)\qquad \oint_\gamma \tilde{S}_{\mathrm{odd}}(x,a,A,B,\eta)dx = 2\pi i\sqrt{a}\Lambda(a,\Gamma(a,A,B,\eta),\eta)\eta + \pi i,$$

where the infinite series $\Lambda(a,\Gamma,\eta)$ *and* $\Gamma(a,A,B,\eta)$ *are those given in Proposition 1.1 and Theorem 1.2 respectively and* γ *is a closed curve that encircles two simple poles counterclockwise.*

Let us now employ the relation (1.141) between $\phi_\pm$ and $\psi_\pm$ to deduce analytic properties of $\psi_{\pm,B}$ from those of $\phi_{\pm,B}$. Here we make full use of microlocal analysis, which has been made possible by the estimation (1.142) that Λ_n satisfies. The concrete procedure is as follows: first, by the Taylor expansion, the right-hand side of (1.141), can be written as

$$\sum_{k=0}^{\infty} \frac{\tilde{\Lambda}^k(a,\Gamma,\eta)}{k!} \frac{\partial^k}{\partial \Lambda^k} \phi_\pm(z,a,\Lambda_0(a,\Gamma),\mu(a),\nu(a),\eta), \tag{3.3}$$

where

$$\tilde{\Lambda}(a,\Gamma,\eta) = \Lambda(a,\Gamma,\eta) - \Lambda_0(a,\Gamma). \tag{3.4}$$

Then, taking into account the estimates (1.142) of Λ_n, we can rewrite (3.3) in the form of an action of a microdifferential operator

$$\mathcal{L} =: \exp\left(\tilde{\Lambda}\theta_\Lambda\right) : \tag{3.5}$$

upon $\phi_{\pm,B}$ through the Borel transformation. Here $: \cdot :$ designates the normal ordered product (cf.[A1]) and θ_Λ is the symbol of ∂_Λ, i.e., $:\theta_\Lambda := \partial_\Lambda$. More concretely, we can write the action of $\mathcal{L}$ as an action of an integro-differential operator so that (3.3) can be rewritten as follows:

Proposition 3.2. *Suppose that the constants $a \neq 0$ and Λ in (1.2) are different from 0. Let $\phi_{\pm,B}$ (resp., $\psi_{\pm,B}$) be the Borel transformed WKB solutions of (1.2) (resp., (1.4)) and suppose that they are both normalized at a simple pole $z = a$. Then they satisfy the following relation:*

$$\begin{aligned} &\psi_{\pm,B}(z,a,\Gamma,y) \\ &\quad = \int_{\mp y_+}^{y} K_\Lambda(a,\Gamma,y-y',\partial_\Lambda)\phi_{\pm,B}(z,a,\Lambda,\mu(a),\nu(a),y')dy'\Big|_{\Lambda=\Lambda_0(a,\Gamma)}, \end{aligned} \tag{3.6}$$

where $K_\Lambda(a,\Gamma,y,\partial_\Lambda)$ is a differential operator of infinite order that is defined on $\{(\Lambda,y) \in \mathbb{C}^2\}$, which analytically depends on a and Γ with the exception $a\Gamma = 0$, and

$$y_+(z,a,\Gamma) = \int_a^z \sqrt{\frac{a\Gamma}{z^2-a^2}}dz, \tag{3.7}$$

$$\Lambda_0(a,\Gamma) = \sqrt{\Gamma}. \tag{3.8}$$

Here $\mu(a)$ and $\nu(a)$ are functions that are respectively given by (1.135) and (1.136).

See [K] and [SKK] for the notion of differential operators of infinite order.

Remark 3.1. The differential operator K_Λ is locally defined for $a, \Gamma \neq 0$. However, as (1.134) implies, K_Λ is multivalued on $\{(a, \Gamma, \Lambda, y) \in \mathbb{C}^4 : a, \Gamma \neq 0\}$.

Remark 3.2. It immediately follows from (3.7) and (3.8) that

$$y_+(z, a, \Gamma) = \int_a^z \sqrt{\frac{a\Lambda_0^2(a, \Gamma)}{z^2 - a^2}} dz. \tag{3.9}$$

Therefore, comparing (2.12) and (3.9), we find that y_+ is preserved by a change of parameters from (Λ, μ, ν) to $(\Gamma, g_+(a), g_-(-a))$.

Combining Theorem 2.1 and Proposition 3.2, we obtain the following

Lemma 3.1. *Let $\psi_\pm(z, a, \Gamma, \eta)$ denote the WKB solutions of the Legendre equation* (1.4) *that are normalized at a simple pole $z = a$ as in* (1.131)*. Then their Borel transform $\psi_{\pm,B}(z, a, \Gamma, y)$ are singular at*

$$y = \mp y_+(z, a, \Gamma) + 2m\pi i\sqrt{a\Gamma} \quad (m = 0, \pm 1, \pm 2, \cdots), \tag{3.10}$$

where $y_+(z)$ is the function given by (3.7)*. Furthermore their alien derivatives there satisfy the following relation* (3.11) *on the condition that z can be connected with $z = \infty$ by a path that is contained in the interior of a Stokes region of the Legendre equation* (1.4)*:*

$$\begin{aligned} &\Big(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma}}\ \psi_\pm\Big)_B (z, a, \Gamma, y) \\ &\quad = \pm\ \Xi_m(\mu, \nu)\big(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi_\pm\big)_B(z, a, \Gamma, y - 2m\pi i\sqrt{a\Gamma}), \end{aligned} \tag{3.11}$$

where $\mu = \mu(a)$ and $\nu = \nu(a)$ are functions that are given by (1.135) *and* (1.136) *respectively and $\tilde{\Lambda}(a, \Gamma, \eta)$ is a formal power series given by* (1.134) *and* (3.4)*.*

Proof. From the representation (3.6) of $\psi_{\pm,B}$ and the definition of the alien derivative, we find

$$\begin{aligned} &\big(\Delta_{y=2m\pi i\sqrt{a\Gamma}}\ e^{\mp\eta y_+}\psi_\pm\big)_B (z, a, \Gamma, y) \\ &\quad = \mathcal{L}_{2m\pi i\sqrt{a}\Lambda}\big(\Delta_{y=2m\pi i\sqrt{a}\Lambda}\ \phi_\pm^{(0)}\big)_B(z, a, \Lambda, y)\big|_{\Lambda=\Lambda_0(a,\Gamma)} \end{aligned} \tag{3.12}$$

holds, where $\mathcal{L}_{y_0}$ is the integro-differential operator obtained by taking $y = y_0$ as the end point of integration instead of $y = \mp y_+$ in (3.6) and $\phi_{\pm}^{(0)} = e^{\mp\eta y_+}\phi_{\pm}$. Therefore it follows from Theorem 2.1 that the right hand side of (3.12) is equal to

$$(3.13) \qquad \pm\, \Xi_m(\mu(a), \nu(a))\mathcal{L}_{2m\pi i\sqrt{a}\Lambda}\big(\phi_{\pm,B}^{(0)}(z, a, \Lambda, y - 2m\pi i\sqrt{a}\Lambda)\big)\big|_{\Lambda=\Lambda_0(a,\Gamma)}.$$

Let us introduce the following coordinate transformation from (y, Λ) to (y', Λ'):

$$(3.14) \qquad \begin{cases} y' = y - 2m\pi i\sqrt{a}\Lambda \\ \Lambda' = \Lambda. \end{cases}$$

We now prepare the following general lemma:

Lemma 3.2. *Let $F : (y, \Lambda_1, \cdots, \Lambda_p) \to (y', \Lambda_1', \cdots, \Lambda_p')$ be a coordinate transformation given by*

$$(3.15) \qquad \begin{cases} y' = y + f(\Lambda_1, \cdots, \Lambda_p) \\ \Lambda_1' = \Lambda_1 \\ \qquad \vdots \\ \Lambda_p' = \Lambda_p, \end{cases}$$

where $f(\Lambda_1, \cdots, \Lambda_p)$ is a holomorphic function of $\Lambda = (\Lambda_1, \cdots, \Lambda_p) \in \mathbb{C}^p$ at $\Lambda = \mathring{\Lambda}$. Let $\tilde{\Lambda}_1, \cdots, \tilde{\Lambda}_p$ be symbols of microdifferential operators of the following form:

$$(3.16) \qquad \tilde{\Lambda}_j(\Lambda_1, \cdots, \Lambda_p, \eta) = \sum_{n=1}^{\infty} \eta^{-n}\Lambda_{j,n}(\Lambda_1, \cdots, \Lambda_p) \quad (j = 1, \cdots, p).$$

Then the following relation holds:

$$(3.17) \qquad \begin{aligned} &: \exp\big(\tilde{\Lambda}(\Lambda, \eta)\cdot\theta_\Lambda\big) : \\ &=: \exp\big[\eta'\big(f(\Lambda' + \tilde{\Lambda}) - f(\Lambda')\big)\big] :: \exp\big(\tilde{\Lambda}(\Lambda', \eta')\cdot\theta_{\Lambda'}\big) :, \end{aligned}$$

where $\eta' = \sigma(\partial/\partial y')$, $\theta_{\Lambda_1} = \sigma(\partial/\partial\Lambda_1), \cdots$, $\theta_\Lambda = (\theta_{\Lambda_1}, \cdots, \theta_{\Lambda_p})$, etc., and $\cdot$ is the inner product.

Proof. Let $P(\Lambda,\theta_\Lambda,\eta)$ denote the symbol of the left-hand side of (3.17), i.e.,

$$P(\Lambda,\theta_\Lambda,\eta)=\exp\big(\tilde\Lambda(\Lambda,\eta)\cdot\theta_\Lambda\big). \tag{3.18}$$

We first note the following equality:

$$:P(\Lambda,\theta_\Lambda,\eta):=:P'(\Lambda',\theta_{\Lambda'},\eta'):, \tag{3.19}$$

where P' is given by

$$\begin{aligned}
&P'(\Lambda',\theta_{\Lambda'},\eta') \\
&=\exp\big[-F(y,\Lambda)\cdot(\eta',\theta_{\Lambda'})\big]\exp(\partial_{\hat\eta}\cdot\partial_{\hat y}+\partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda}) \\
&\quad\times P(\Lambda,\hat\theta_\Lambda,\hat\eta)\exp\big[F(\hat y,\hat\Lambda)\cdot(\eta',\theta_{\Lambda'})\big]\Big|_{\substack{(y,\Lambda)=(\hat y,\hat\Lambda)=F^{-1}(y',\Lambda')\\ \hat\theta_\Lambda=\hat\eta=0}} \\
&=\exp(\partial_{\hat\eta}\cdot\partial_{\hat y}+\partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})P(\Lambda,\hat\theta_\Lambda,\hat\eta) \\
&\quad\times\exp\big[(F(y+\hat y,\Lambda+\hat\Lambda)-F(y,\Lambda))\cdot(\eta',\theta_{\Lambda'})\big]\Big|_{\substack{(y,\Lambda)=F^{-1}(y',\Lambda')\\ \hat\Lambda=\hat y=\hat\theta_\Lambda=\hat\eta=0}}.
\end{aligned} \tag{3.20}$$

(Cf. [SKK, Chapter 2, Theorem 1.5.5]. See also the proof of [AKY, Proposition 1.2.13].) Since

$$\begin{aligned}
&(F(y+\hat y,\Lambda+\hat\Lambda)-F(y,\Lambda))\cdot(\eta',\theta_{\Lambda'}) \\
&=\hat y\eta'+(f(\Lambda+\hat\Lambda)-f(\Lambda))\eta'+\hat\Lambda\cdot\theta_{\Lambda'}
\end{aligned} \tag{3.21}$$

and

$$e^{-\hat z\zeta}\exp(\partial_{\hat\zeta}\cdot\partial_{\hat z})e^{\hat z\zeta}f(\hat\zeta)=f(\hat\zeta+\zeta) \tag{3.22}$$

holds for a holomorphic function $f(\zeta)$, we find

$$\begin{aligned}
&P'(\Lambda',\theta_{\Lambda'},\eta') \\
&=\exp(\partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})P(\Lambda,\hat\theta_\Lambda,\hat\eta+\eta') \\
&\quad\times\exp\big[(f(\Lambda+\hat\Lambda)-f(\Lambda))\eta'+\hat\Lambda\cdot\theta_{\Lambda'}\big]\Big|_{\substack{(y,\Lambda)=F^{-1}(y',\Lambda')\\ \hat\Lambda=\hat y=\hat\theta_\Lambda=\hat\eta=0}} \\
&=\exp(\partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})\exp\big(\tilde\Lambda_1(\Lambda,\eta')\hat\theta_{\Lambda_1}\big)\cdots\exp\big(\tilde\Lambda_p(\Lambda,\eta')\hat\theta_{\Lambda_p}\big) \\
&\quad\times\exp\big[(f(\Lambda+\hat\Lambda)-f(\Lambda))\eta'+\hat\Lambda\cdot\theta_{\Lambda'}\big]\Big|_{\substack{\Lambda=\Lambda'\\ \hat\Lambda=\hat\theta_\Lambda=0}}
\end{aligned} \tag{3.23}$$

$$= \exp\big[\eta'\big(f(\Lambda'+\tilde{\Lambda})-f(\Lambda')\big)+\tilde{\Lambda}\cdot\theta_{\Lambda'}\big].$$

Thus we obtain (3.17) from (3.19).

□

We resume the proof of Lemma 3.1. It follows from (3.17) that

$$\begin{aligned}(3.24)\qquad \mathcal{L} &=: \exp\big(\tilde{\Lambda}(a,\Gamma,\eta)\theta_{\Lambda}\big):\\ &=: \exp\big(-2m\pi i\sqrt{a}\eta'\tilde{\Lambda}(a,\Gamma,\eta')\big) :: \exp\big(\tilde{\Lambda}(a,\Gamma,\eta')\theta_{\Lambda'}\big):.\end{aligned}$$

Therefore we find

$$\begin{aligned}(3.25)\qquad &\mathcal{L}_{2m\pi i\sqrt{a}\Lambda}\big(\phi^{(0)}_{\pm,B}(z,a,\Lambda,y-2m\pi i\sqrt{a}\Lambda)\big)\\ &=: \exp\big(-2m\pi i\sqrt{a}(\Lambda_1+\Lambda_2\eta'^{-1}+\cdots)\big):\big(\mathcal{L}_0\phi^{(0)}_{\pm,B}\big)(z,a,\Lambda',y'),\end{aligned}$$

where the action of $:\eta'^{-1}:$ is fixed by taking $y'=0$ as the end point of integration. Here we note that, from (3.6) and (3.8), we obtain

$$\begin{aligned}(3.26)\qquad &\big(\mathcal{L}_0\phi^{(0)}_{\pm,B}\big)(z,a,\Lambda,y-2m\pi i\sqrt{a}\Lambda)\big|_{\Lambda=\Lambda_0(a,\Gamma)}\\ &= \big(e^{\mp\eta y_\pm}\psi_\pm\big)_B(z,a,\Gamma,y-2m\pi i\sqrt{a\Gamma}).\end{aligned}$$

Then (3.11) follows from (3.12), (3.13), (3.25) and (3.26).

□

From (3.1) and (3.8), we find that (3.11) can be rewritten as follows:

$$\begin{aligned}(3.27)\qquad &\Big(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma}}\ \psi_\pm\Big)_B(z,a,\Gamma,y)\\ &= \pm(-1)^m\Xi_m(\mu,\nu)\big(\exp(-m\oint_\gamma S_{\mathrm{odd}}dx)\psi_\pm\big)_B(z,a,\Gamma,y).\end{aligned}$$

Now, we will study the singularity structure of Borel transformed WKB solutions of the Mathieu equation (1.1) using the transformation obtained in Theorem 1.1. To begin with, to simplify the notation, we restate the estimates (1.12) and (1.13) in the following form: there exists

(3.28) a continuous increasing function $h:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ that satisfies $h(\delta)\to 0$ when $\delta\to 0$

such that z_{2n} and Γ_{2n} ($n \geq 1$) given in Theorem 1.1 satisfy the following estimates on $E^2_{r_1,\delta}$ for $0 < \delta < r_2$:

$$|z_{2n}(x, a, A, B)| \leq (2n)! h^n(\delta) |aA|^{-n}, \tag{3.29}$$

$$|\Gamma_{2n}(a, A, B)| \leq (2n)! h^n(\delta) |aA|^{-n}. \tag{3.30}$$

Let us consider the following ∞-Legendre equation:

$$\left(\frac{d^2}{dz^2} - \eta^2 \left(\frac{a\Gamma(a, A, B, \eta)}{z^2 - a^2} + \eta^{-2} \left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2} \right) \right) \right) \psi^\dagger = 0. \tag{3.31}$$

We immediately see that WKB solutions $\psi^\dagger_\pm(z, a, A, B, \eta)$ of (3.31) that are normalized at its simple pole $z = a$ are given by

$$\psi^\dagger_\pm(z, a, A, B, \eta) = \psi_\pm(z, a, \Gamma(a, A, B, \eta), \eta). \tag{3.32}$$

Similarly to the relation between $\phi_{\pm,B}$ and $\psi_{\pm,B}$ discussed in Proposition 3.2, by applying the Taylor expansion and the Borel transformation successively to (3.32), we can relate the Borel transform of $\psi^\dagger_\pm$ with that of $\psi_\pm$ through the action of a microdifferential operator defined by

$$\mathcal{G} =: \exp(\tilde{\Gamma}\theta_\Gamma) :, \tag{3.33}$$

where

$$\tilde{\Gamma}(a, A, B, \eta) = \Gamma(a, A, B, \eta) - \Gamma_0(a, A, B) \tag{3.34}$$

and θ_Γ is the symbol of ∂_Γ. To be more specific, we find the following thanks to (3.30):

Proposition 3.3. *Let $\psi_{\pm,B}$ (resp. $\psi^\dagger_{\pm,B}$) be the Borel transformed WKB solutions of* (1.4) *(resp.* (3.31)*) for $a \neq 0$ (resp. $A \neq 0$) that are normalized at a simple pole $z = a$. Then $\psi_{\pm,B}$ and $\psi^\dagger_{\pm,B}$ satisfy the following relation:*

$$\begin{aligned} &\psi^\dagger_{\pm,B}(z, a, A, B, y) \\ &\quad = \int_{\mp y_+}^{y} K_\Gamma(a, A, B, y - y', \partial_\Gamma) \psi_{\pm,B}(z, a, \Gamma, y') dy' \Big|_{\Gamma = \Gamma_0(a,A,B)}, \end{aligned} \tag{3.35}$$

where $K_\Gamma(a, A, B, y, \partial_\Gamma)$ is a differential operator of infinite order that is defined on

$$\{(a, A, B, \Gamma, y) \in \mathbb{C}^5 : a, A \neq 0, |B/A| < r_2, h(|B/A|)|y| < \sqrt{|aA|}\}, \tag{3.36}$$

$$(3.37) \qquad y_+(z,a,A,B) = \int_a^z \sqrt{\frac{a\Gamma_0(a,A,B)}{z^2-a^2}}\,dz,$$

and

$$(3.38) \qquad \sqrt{\Gamma_0(a,A,B)} = \frac{1}{2\pi i\sqrt{a}}\int_\gamma \sqrt{\frac{aA+xB}{x^2-a^2}}\,dx.$$

In view of Lemma 3.1, we expect that $\psi^\dagger_{\pm,B}$ have singularities at $y = \mp y_+ + 2m\pi i\sqrt{a\Gamma_0}$ $(m = 0, \pm 1, \pm 2, \cdots)$. This is the case if the representation (3.35) holds there, that is, they actually have the singularities there that correspond to those of $\psi_{\pm,B}$. Let us confirm this fact when these singularities are contained in the domain of definition of the integro-differential operator given in Proposition 3.3. We first note that Γ_0 is independent of a. Indeed, by taking $\tilde{x} = x/a$ as a new variable, we obtain

$$(3.39) \qquad \sqrt{\Gamma_0(a,A,B)} = \frac{1}{2\pi i}\int_\gamma \sqrt{\frac{A+\tilde{x}B}{\tilde{x}^2-1}}\,d\tilde{x}.$$

Therefore, by taking r_2 sufficiently small, we can assume that

$$(3.40) \qquad \frac{1}{2}|A| < \big|\Gamma_0(a,A,B)\big| < 2|A|$$

holds on $\{|B| < r_2|A|\}$. Hence, if $m \in \mathbb{Z}$, A and B satisfy

$$(3.41) \qquad 2\sqrt{2}|m|\pi h(|B/A|) < 1,$$

the m-th singular point is in the domain of definition of the integro-differential operator. For each $m \in \mathbb{Z}$, this condition is satisfied by taking $|B/A|$ sufficiently small. Further, through the representation (3.35), we can derive from Lemma 3.1 the following

Lemma 3.3. *Let $\psi^\dagger_\pm(z,a,A,B,\eta)$ denote the WKB solutions of the ∞-Legendre equation* (3.31) *that are normalized at a simple pole $z = a$. Then, when* (3.41) *holds, its Borel transform $\psi^\dagger_{\pm,B}(z,a,A,B,\ y)$ is singular at*

$$(3.42) \qquad y = \mp y_+(z,a,A,B) + 2m\pi i\sqrt{a\Gamma_0(a,A,B)}$$

and its alien derivative there satisfies

$$(3.43) \qquad \left(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma_0}}\ \psi^\dagger_\pm\right)_B(z,a,A,B,y)$$

$$= \pm(-1)^m \Xi_m(\mu,\nu)\big(\exp(-m\oint_\gamma S_{\rm odd}^\dagger dx)\psi_\pm^\dagger\big)_B(z,a,A,B,y),$$

where $\mu = \mu(a)$ *and* $\nu = \nu(a)$ *are functions that are given by* (1.135) *and* (1.136) *respectively and* $S_{\rm odd}^\dagger$ *is the odd part of the solutions of the Riccati equation associated with* (3.31)*.*

Proof. As in the proof of Lemma 3.1, it suffices to show

$$\mathcal{G}_{2m\pi i\sqrt{a\Gamma}}\big(\big(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi_\pm^{(0)}\big)_B(z,a,\Gamma,y-2m\pi i\sqrt{a\Gamma})\big)\big|_{\Gamma=\Gamma_0} \tag{3.44}$$
$$= \big(\exp(-m\oint_\gamma S_{\rm odd}^\dagger dx)e^{\mp\eta y_+}\psi_\pm^\dagger\big)_B(z,a,A,B,y),$$

where $\mathcal{G}_{y_0}$ is the integro-differential operator obtained by taking $y = y_0$ as the end point of integration instead of $y = \mp y_+$ in (3.35) and $\psi_\pm^{(0)} = e^{\mp\eta y_+}\psi_\pm$. Let us introduce the following coordinate transformation from (y,Γ) to (y',Γ'):

$$\begin{cases} y' = y - 2m\pi i\sqrt{a\Gamma} \\ \Gamma' = \Gamma. \end{cases} \tag{3.45}$$

Then, from Lemma 3.2, we obtain

(3.46)
$$\mathcal{G} =: \exp\big(\tilde{\Gamma}(a,A,B,\eta)\theta_\Gamma\big):$$
$$=: \exp\big[-2m\pi i\sqrt{a}\eta'\big(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'}\big)\big] :: \exp\big(\tilde{\Gamma}(a,A,B,\eta')\theta_{\Gamma'}\big):.$$

Therefore we find

(3.47)
$$\mathcal{G}_{2m\pi i\sqrt{a\Gamma}}\big(\big(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi_\pm^{(0)}\big)_B(z,a,\Gamma,y-2m\pi i\sqrt{a\Gamma})\big)$$
$$=: \exp\big[-2m\pi i\sqrt{a}\eta'\big(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'}\big)\big]:$$
$$\times : \exp\big(\tilde{\Gamma}(a,A,B,\eta')\theta_{\Gamma'}\big) : \big(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta')\psi_\pm^{(0)}\big)_B(z,a,\Gamma',y')$$
$$=: \exp\big[-2m\pi i\sqrt{a}\eta'\big(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'}\big)\big]:$$
$$\times \big(\exp(-2m\pi i\sqrt{a}\eta'\tilde{\Lambda}(a,\Gamma'+\tilde{\Gamma},\eta'))\psi_\pm^{(0)}(z,a,\Gamma'+\tilde{\Gamma},\eta')\big)_B$$

where the action of $: \eta'^{-1} :$ is fixed by taking $y' = 0$ as the end point of integration. From (3.8) and (3.32), we find that, by replacing Γ with $\Gamma_0(a, A, B)$, the rightmost term of (3.47) equals to

$$: \exp\big[-2m\pi i\sqrt{a}\eta\big(\Lambda(a, \Gamma(a, A, B, \eta), \eta) - \sqrt{\Gamma_0}(a, A, B)\big)\big] : \times \big(e^{\mp\eta y_+}\psi^{\dagger}_{\pm}\big)_B\big(z, a, A, B, y - 2m\pi i\sqrt{a\Gamma_0}\big). \tag{3.48}$$

Then (3.43) follows from the following equality:

$$\oint_{\gamma} S^{\dagger}_{\text{odd}}(z, a, A, B, \eta)dz = 2\pi i\sqrt{a}\Lambda(a, \Gamma(a, A, B, \eta), \eta)\eta + \pi i. \tag{3.49}$$

□

Now, we derive the singularity structure of Borel transformed WKB solutions of the Mathieu equation (1.1) from Lemma 3.3. We first remark that the Mathieu equation has two simple poles and one simple turning point. On the other hand, the (∞-)Legendre equation has only two simple poles. Therefore, if we want to relate the Mathieu equation with the Legendre equation, in other words, if we want to focus our attention on the two simple poles of the Mathieu equation, we have to remove the effect of the simple turning point. This can be attained by controlling the merging velocity of the turning point, that is, $|A/B|$. Indeed, since the turning point is located at $x = -aA/B$, it is distant enough from the poles located at $x = \pm a$ if $|A/B|$ is large. The existence of the function $h(\delta)$ that satisfies (3.28) $\sim$ (3.30) enables us to ignore the effect of the simple turning point and to derive the structure of Borel transformed WKB solutions of the Mathieu equation at the (fixed) singularities related only to the two simple poles from that of the Legendre equation as is discussed below (especially in Theorem 3.2).

Let $\tilde{\psi}_{\pm}$ be WKB solutions of the Mathieu equation (1.1). Then, from (1.132), we obtain the following relation:

$$\tilde{\psi}_{\pm}(x, a, A, B, \eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2} \psi^{\dagger}_{\pm}(z(x, a, A, B, \eta), a, A, B, \eta). \tag{3.50}$$

For the simplicity of discussion, we take $z_0(x, a, A, B)$ as a new coordinate variable instead of x. This is guaranteed by Theorem 1.1. Let M and L_{∞} respectively be the Borel transformed Mathieu operator expressed in

(z_0, a, A, B, y)-coordinate and the Borel transformed ∞-Legendre operator, i.e.,

$$M = \left(\frac{\partial x}{\partial z_0}\right)^{-2} \frac{\partial^2}{\partial z_0^2} - \frac{\partial^2 x}{\partial z_0^2}\left(\frac{\partial x}{\partial z_0}\right)^{-3} \frac{\partial}{\partial z_0} - \frac{aA + xB}{x^2 - a^2}\frac{\partial^2}{\partial y^2} - \frac{g_+(a)}{(x-a)^2} - \frac{g_-(-a)}{(x+a)^2}, \tag{3.51}$$

$$L_\infty = \frac{\partial^2}{\partial z_0^2} - \frac{a\Gamma(a, A, B, \partial/\partial y)}{z_0^2 - a^2}\frac{\partial^2}{\partial y^2} - \frac{g_+(a)}{(z_0-a)^2} - \frac{g_-(-a)}{(z_0+a)^2}. \tag{3.52}$$

Then, we find the following

Theorem 3.1. *There exist invertible microdifferential operators $\mathcal{Z}$ and $\mathcal{W}$ that satisfy*

$$M\mathcal{Z} = \mathcal{W} L_\infty \tag{3.53}$$

on

$$\left\{(z_0, y, a, A, B; \zeta_0, \eta) \in T^*\mathbb{C}_{z_0} \times \dot{T}^*\mathbb{C}_y \times \mathbb{C}^3 : |z_0| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\right\} \tag{3.54}$$

for some positive constants r_1 and r_2 with the exception of $z_0^2 - a^2 = 0$. The concrete form of $\mathcal{Z}$ and $\mathcal{W}$ are as follows:

$$\mathcal{Z} =: \left(\frac{\partial x}{\partial z_0}\right)^{1/2}\left(1 + \frac{\partial \tilde{z}}{\partial z_0}\right)^{-1/2} \exp\left(\tilde{z}(z_0, a, A, B, \eta)\zeta_0\right) :, \tag{3.55}$$

$$\mathcal{W} =: \left(\frac{\partial x}{\partial z_0}\right)^{-3/2}\left(1 + \frac{\partial \tilde{z}}{\partial z_0}\right)^{3/2} \exp\left(\tilde{z}(z_0, a, A, B, \eta)\zeta_0\right) :, \tag{3.56}$$

where

$$\tilde{z}(z_0, a, A, B, \eta) = z(x(z_0, a, A, B), a, A, B, \eta) - z_0. \tag{3.57}$$

Theorem 3.1 follows from the following proposition (cf. [AY]):

Proposition 3.4. *Let $x(t)$ be a holomorphic change of variables at the origin from $\mathbb{C}_t$ to $\mathbb{C}_x$ satisfying*

$$x(0) = 0 \text{ and } \frac{dx}{dt}(0) \neq 0 \tag{3.58}$$

and suppose that the following microdifferential operators $\mathcal{P}$ and $\mathcal{Q}$ are given:

$$\mathcal{P} = \frac{\partial^2}{\partial t^2} - p\Big(t, \frac{\partial}{\partial y}\Big)\frac{\partial^2}{\partial y^2}, \tag{3.59}$$

$$\mathcal{Q} = \frac{\partial^2}{\partial x^2} - q\Big(x, \frac{\partial}{\partial y}\Big)\frac{\partial^2}{\partial y^2}, \tag{3.60}$$

where p (resp., q) are microdifferential operators of order 0 defined near $t = 0$ (resp., $x = 0$) except for $\eta = 0$. Furthermore let $r(x, \eta)$ be the symbol of a microdifferential operator of order -1 and suppose that the total symbols $p(t, \eta) := \sigma\big(p(t, \partial/\partial y)\big)$, $q(x, \eta) := \sigma\big(q(x, \partial/\partial y)\big)$ and $z(x, \eta) = x + r(x, \eta)$ satisfy the following relation:

$$p(t, \eta) = \Big(\frac{dz(x(t), \eta)}{dt}\Big)^2 q(z(x(t), \eta), \eta) - \frac{1}{2}\eta^{-2}\{z(x(t), \eta); t\}. \tag{3.61}$$

Then the following relation holds:

$$\mathcal{P}\mathcal{X} = \mathcal{Y}\mathcal{Q}, \tag{3.62}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are microdifferential operators defined by

$$\mathcal{X} =: \Big(\frac{dz}{dt}\Big)^{-1/2} \exp\big(r(x(t), \eta)\xi\big) :, \tag{3.63}$$

$$\mathcal{Y} =: \Big(\frac{dz}{dt}\Big)^{3/2} \exp\big(r(x, \eta)\xi\big) : \tag{3.64}$$

and $\xi = \sigma(\partial/\partial x)$.

Proof. Let $P(x, \xi, \eta)$, $Q(x, \xi, \eta)$, $X(x, \xi, \eta)$ and $Y(x, \xi, \eta)$ respectively be total symbols of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{X}$ and $\mathcal{Y}$ in (x, y)-coordinate. For example, $P(x, \xi, \eta)$ and $Q(x, \xi, \eta)$ are respectively given by

$$P(x, \xi, \eta) = \Big(\frac{dt}{dx}\Big)^{-2}\xi^2 - \Big(\frac{dt}{dx}\Big)^{-3}\frac{d^2 t}{dx^2}\xi - \eta^2 p(t(x), \eta) \tag{3.65}$$

and

$$Q(x, \xi, \eta) = \xi^2 - \eta^2 q(x, \eta), \tag{3.66}$$

where $t(x)$ is the inverse function of $x(t)$. Then it suffices to show

$$P \circ X(x,\xi,\eta) = Y \circ Q(x,\xi,\eta), \tag{3.67}$$

where the composition $\circ$ is defined by

$$P \circ X(x,\xi,\eta) = \exp\left(\partial_{\hat{\xi}}\partial_{\hat{x}}\right) P(x,\hat{\xi},\eta)X(\hat{x},\xi,\eta)\Big|_{\substack{\hat{x}=x\\ \hat{\xi}=\xi}}. \tag{3.68}$$

(Cf. [A2, Proposition 2.5].) We first note that $P(x,\xi,\eta)$ is expressed in terms of the total symbol

$$\tilde{P}(t,\tau,\eta) = \tau^2 - \eta^2 p(t,\eta) \tag{3.69}$$

of $\mathcal{P}$ in (t,y)-coordinate, where $\tau = \sigma(\partial/\partial t)$, as follows:

$$P(x,\xi,\eta) = e^{-x\xi}\exp\left(\partial_{\hat{\tau}}\partial_{\hat{t}}\right)\tilde{P}(t,\hat{\tau},\eta)e^{x(\hat{t})\xi}\Big|_{\substack{\hat{t}=t(x)\\ \hat{\tau}=0}}. \tag{3.70}$$

Combining (3.68) and (3.70), we find

$$\begin{aligned} P \circ X(x,\xi,\eta) &\\ &= \exp\left(\partial_{\hat{\tau}}\partial_{\hat{t}}\right)\tilde{P}(t,\hat{\tau},\eta)\exp\left(\partial_{\hat{\xi}}\partial_{\hat{x}}\right)e^{(x(\hat{t})-x)\hat{\xi}}X(\hat{x},\xi,\eta)\Big|_{\substack{\hat{t}=t(x),\hat{\tau}=0\\ \hat{x}=x,\hat{\xi}=\xi}} \\ &= \exp\left(\partial_{\hat{\tau}}\partial_{\hat{t}}\right)\tilde{P}(t,\hat{\tau},\eta)e^{(x(\hat{t})-x)\xi}X(x(\hat{t}),\xi,\eta)\Big|_{\substack{\hat{t}=t(x)\\ \hat{\tau}=0}}. \end{aligned} \tag{3.71}$$

Therefore it follows from the concrete form of $\tilde{P}(t,\tau,\eta)$ and (3.71) that

$$\begin{aligned} P \circ X(x,\xi,\eta) = &-\eta^2 p(t(x),\eta)X(x,\xi,\eta) \\ &+ \frac{\partial^2}{\partial\hat{t}^2}\left(e^{(x(\hat{t})-x)\xi}X(x(\hat{t}),\xi,\eta)\right)\Big|_{\hat{t}=t(x)}. \end{aligned} \tag{3.72}$$

On the other hand, since Y satisfies $\partial_\xi^k Y = r^k(x,\eta)Y$, we find

$$\begin{aligned} Y \circ Q(x,\xi,\eta) &\\ &= Y(x,\xi,\eta)Q(x,\xi,\eta) - \eta^2\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^k}{\partial\xi^k}Y(x,\xi,\eta)\frac{\partial^k}{\partial x^k}q(x,\eta) \end{aligned} \tag{3.73}$$

$$\begin{aligned}
&= Y(x,\xi,\eta)Q(x,\xi,\eta) + \eta^2 Y(x,\xi,\eta)q(x,\eta) \\
&\quad - \eta^2 Y(x,\xi,\eta)q(x+r(x,\eta),\eta) \\
&= Y(x,\xi,\eta)\xi^2 - \eta^2 Y(x,\xi,\eta)q(z(x,\eta),\eta).
\end{aligned}$$

Then, since $Y = (dz/dt)^2 X$, it follows from (3.61) and (3.73) that

$$\begin{aligned}
Y \circ Q(x,\xi,\eta) = {} & Y(x,\xi,\eta)\xi^2 - \eta^2 X(x,\xi,\eta)p(t(x),\eta) \\
& - \frac{1}{2}\{z;t\}X(x,\xi,\eta).
\end{aligned} \tag{3.74}$$

Thus, comparing (3.72) and (3.74), we find that (3.67) immediately follows if the following relation is confirmed:

$$\frac{\partial^2}{\partial \hat{t}^2}\Big(e^{(x(\hat{t})-x)\xi}X(x(\hat{t}),\xi,\eta)\Big)\Big|_{\hat{t}=t(x)} = Y(x,\xi,\eta)\xi^2 - \frac{1}{2}\{z;t\}X(x,\xi,\eta). \tag{3.75}$$

Since the left hand side of (3.75) is equal to

$$\begin{aligned}
& e^{-x\xi}\frac{\partial^2}{\partial \hat{t}^2}\bigg(\Big(\frac{dz(x(\hat{t}),\eta)}{d\hat{t}}\Big)^{-1/2}\exp\big(z(x(\hat{t}),\eta)\xi\big)\bigg)\bigg|_{\hat{t}=t(x)} \\
&= \exp\big(r(x,\eta)\xi\big)\frac{\partial^2}{\partial t^2}\Big(\frac{dz}{dt}\Big)^{-1/2} + \Big(\frac{dz}{dt}\Big)^{3/2}\xi^2\exp\big(r(x,\eta)\xi\big),
\end{aligned} \tag{3.76}$$

we find that (3.75) is an immediate consequence of

$$\{z;t\} = -2\Big(\frac{dz}{dt}\Big)^{1/2}\frac{d^2}{dt^2}\Big(\frac{dz}{dt}\Big)^{-1/2}. \tag{3.77}$$

This completes the proof.

□

Remark 3.3. In the situation of Theorem 3.1, $\mathcal{P}$ and $\mathcal{Q}$ correspond to M and L_∞ respectively.

In view of (3.29), we obtain the following

Proposition 3.5. *Let $\psi_{\pm,B}$ and $\tilde{\psi}_{\pm,B}$ respectively be the Borel transformed WKB solutions of* (1.4) *and* (1.1) *for $a \neq 0$ and $A \neq 0$ that are normalized at their simple poles as* (1.131) *and* (1.130). *Then they satisfy the following relation:*

$$\tilde{\psi}_{\pm,B}(z_0,a,A,B,y) \tag{3.78}$$

$$= \int_{\mp y_+}^{y} K_z(z_0, a, A, B, y - y', \partial_{z_0})\psi_{\pm,B}(z_0, a, A, B, y')dy',$$

where $K_z(z_0, a, A, B, y, \partial_{z_0})$ *is a differential operator of infinite order that is defined on*

$$\tilde{E}^2_{r_1,h} = \{(z_0, a, A, B, y) \in \mathbb{C}^5 : a, A \neq 0, |x| < r_1|a|, |B/A| < r_2, \quad h(|B/A|)|y| < \sqrt{|aA|}\} \tag{3.79}$$

with some positive constants $r_1 > 1$ *and* $r_2 > 0$ *and*

$$y_+(z_0, a, A, B) = \int_a^{z_0} \sqrt{\frac{a\Gamma_0(a, A, B)}{z_0^2 - a^2}} dz_0. \tag{3.80}$$

In conclusion, by employing similar discussions to Lemma 3.3 and Proposition 3.1, we obtain

Theorem 3.2. *Let* $\tilde{\psi}_\pm(x, a, A, B, \eta)$ *be WKB solutions of the Mathieu equation (1.1) with* $a \neq 0$ *and* $A \neq 0$ *that is normalized at a simple pole* $x = a$*. Then, for each integer* m *we can take some positive constant* δ *so that the following holds when* $|B/A| < \delta$ *is satisfied: The Borel transform* $\tilde{\psi}_{\pm,B}(x, a, A, B, y)$ *of* $\tilde{\psi}_\pm(x, a, A, B, \eta)$ *is singular at*

$$y = \mp y_+(x, a, A, B) + 2m\pi i\sqrt{a\Gamma_0(a, A, B)} \tag{3.81}$$

and its alien derivative there satisfies

$$\left(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma_0}}\ \tilde{\psi}_\pm\right)_B (x, a, A, B, y) = \pm(-1)^m \Xi_m(\mu, \nu)\big(\exp(-m\oint_\gamma \tilde{S}_{\mathrm{odd}}dx)\tilde{\psi}_\pm\big)_B(x, a, A, B, y), \tag{3.82}$$

where

$$\Xi_m(\mu, \nu) = \frac{1}{m}\left\{1 + (-1)^m - \cosh\left(2\pi i m\sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}}\right) - \cosh\left(2\pi i m\sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}}\right)\right\}, \tag{3.83}$$

$$\mu = \mu(a) = \sqrt{1 + 2(g_+(a) + g_-(-a))}, \tag{3.84}$$

$$\nu = \nu(a) = 2(g_+(a) - g_-(-a)), \tag{3.85}$$

$$y_+(x, a, A, B) = \int_a^x \sqrt{\frac{aA + xB}{x^2 - a^2}} dx \tag{3.86}$$

and

$$\sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i \sqrt{a}} \int_\gamma \sqrt{\frac{aA + xB}{x^2 - a^2}} dx. \tag{3.87}$$

Here γ *is a closed curve that encircles two simple poles counterclockwise.*

Remark 3.4. In Theorem 3.2, the positive constant δ should be taken so small that (3.41) is satisfied for $|B/A| < \delta$ for an arbitrarily given $m \in \mathbb{Z}$.

Remark 3.5. In (x, a, A, B)-coordinate, $y_+(x, a, A, B)$ is given by (3.86). However, since

$$z_0(x, a, A, B) = a \cos \left(\frac{1}{\sqrt{a\Gamma_0}} \int_a^x \sqrt{\frac{aA + xB}{x^2 - a^2}} dx \right) \tag{3.88}$$

satisfies

$$\frac{aA + xB}{x^2 - a^2} = \left(\frac{\partial z_0}{\partial x} \right)^2 \frac{a\Gamma_0}{z_0^2 - a^2}, \tag{3.89}$$

we find (3.80) is equivalent to (3.86).

4. Analytic properties of Borel transformed WKB solutions of an M2P1T equation

In this section, we study WKB theoretic structure of an M2P1T equation

$$\left(\frac{d^2}{dt^2} - \eta^2 Q(t, a, \rho, \eta) \right) \hat{\psi} = 0, \tag{4.1}$$

where the potential $Q(t, a, \rho)$ is given in [Part I, Definition 1.1]. We constructed transformation series $x(t, a, \rho, \eta)$, $A(a, \rho, \eta)$ and $B(a, \rho, \eta)$ in [Part I, Section 1] that give equivalence between an M2P1T equation and the following ∞-Mathieu equation:

(4.2)
$$\Big(\frac{d^2}{dx^2}-\eta^2\Big(\frac{aA(a,\rho,\eta)+xB(a,\rho,\eta)}{x^2-a^2}+\eta^{-2}\Big(\frac{g_+(a)}{(x-a)^2}+\frac{g_-(-a)}{(x+a)^2}\Big)\Big)\Big)\tilde{\psi}^\dagger=0.$$

As the following discussion shows, (4.2) behaves as the WKB theoretic canonical form of an M2P1T equation.

Let $\hat{\psi}_\pm$ and $\tilde{\psi}^\dagger_\pm$ respectively be WKB solutions of (4.1) and (4.2) that are normalized at their simple poles $t=a$ and $x=a$. Then, from [Part I, Theorem 1.3.3], we find the following relation holds:

$$\hat{\psi}_\pm(t,a,\rho,\eta)=\Big(\frac{\partial x}{\partial t}\Big)^{-1/2}\tilde{\psi}^\dagger_\pm(x(t,a,\rho,\eta),a,\rho,\eta). \tag{4.3}$$

For the simplicity of discussion, we take $x_0(t,a,\rho)$ as a new coordinate variable instead of t. This is guaranteed by [Part I, Theorem 1.3.1]. Let N and M_∞ respectively be the Borel transformed M2P1T operator expressed in (x_0,a,ρ,y)-coordinate and the Borel transformed ∞-Mathieu operator, i.e.,

$$N=\Big(\frac{\partial t}{\partial x_0}\Big)^{-2}\frac{\partial^2}{\partial x_0^2}-\frac{\partial^2 t}{\partial x_0^2}\Big(\frac{\partial t}{\partial x_0}\Big)^{-3}\frac{\partial}{\partial x_0}-Q(t,a,\rho,\partial/\partial y)\frac{\partial^2}{\partial y^2} \tag{4.4}$$

$$\begin{aligned}M_\infty=&\frac{\partial^2}{\partial x_0^2}-\frac{aA(a,\rho,\partial/\partial y)+x_0B(a,\rho,\partial/\partial y)}{x_0^2-a^2}\frac{\partial^2}{\partial y^2}\\&-\frac{g_+(a)}{(x_0-a)^2}-\frac{g_-(-a)}{(x_0+a)^2}.\end{aligned} \tag{4.5}$$

Then, from [Part I, Theorem 1.3.1] and Proposition 3.4, we obtain the following

Theorem 4.1. *There exist invertible microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ that satisfy*

$$N\mathcal{X}=\mathcal{Y}M_\infty \tag{4.6}$$

on

$$\{(x_0,y,a,\rho;\xi_0,\eta)\in T^*\mathbb{C}_{x_0}\times\dot{T}^*\mathbb{C}_y\times\mathbb{C}^2: \tag{4.7}$$

$$|x_0| < r, 0 < |\rho| < r, R_0|a| < |\rho|\}$$

for some positive constants r *and* R_0 *with the exception of* $x_0^2 - a^2 = 0$*. The concrete form of* $\mathcal{X}$ *and* $\mathcal{Y}$ *are as follows:*

$$\mathcal{Z} =: \left(\frac{\partial t}{\partial x_0}\right)^{1/2}\left(1 + \frac{\partial \tilde{x}}{\partial x_0}\right)^{-1/2} \exp\left(\tilde{x}(x_0, a, \rho, \eta)\xi_0\right) :, \tag{4.8}$$

$$\mathcal{W} =: \left(\frac{\partial t}{\partial x_0}\right)^{-3/2}\left(1 + \frac{\partial \tilde{x}}{\partial x_0}\right)^{3/2} \exp\left(\tilde{x}(x_0, a, \rho, \eta)\xi_0\right) :, \tag{4.9}$$

where

$$\tilde{x}(x_0, a, \rho, \eta) = x(t(x_0, a, \rho), a, \rho, \eta) - x_0. \tag{4.10}$$

For the correspondence of Borel transformed WKB solutions, we have the following

Proposition 4.1. *Let* $\hat{\psi}_{\pm,B}$ *and* $\tilde{\psi}^{\dagger}_{\pm,B}$ *respectively be Borel transformed WKB solutions of a generic M2P1T equation (i.e.* $a, \rho \neq 0$*) and the* ∞*-Mathieu equation that are normalized at their simple poles* $t = a$ *and* $x = a$*. Then they satisfy the following relation:*

$$\hat{\psi}_{\pm,B}(x_0, a, \rho, y) = \int_{\mp y_+}^{y} K_x(x_0, a, \rho, y - y', \partial_{x_0})\tilde{\psi}^{\dagger}_{\pm,B}(x_0, a, \rho, y')dy', \tag{4.11}$$

where $K_x(x_0, a, \rho, y, \partial_{x_0})$ *is a differential operator of infinite order that is defined on*

$$\begin{aligned}\tilde{E}^1_{r,R_0,R_1} = \{(x_0, a, \rho, y) \in \mathbb{C}^4 :&|x_0| < r, 0 < |\rho| < r,\\ &R_0|a| < |\rho|, R_1|y| < \sqrt{|\rho|}\},\end{aligned} \tag{4.12}$$

and

$$y_+(x_0, a, \rho) = \int_a^{x_0} \sqrt{\frac{aA(a, \rho) + x_0 B(a, \rho)}{x_0^2 - a^2}} dx_0. \tag{4.13}$$

Thus, the analysis of the singularity structure of Borel transformed WKB solutions of an M2P1T equation should be reduced to that of the ∞-Mathieu equation. However the complete singularity structure of Borel transformed

WKB solutions of the (∞-)Mathieu equation is too complicated to be analyzed directly. Fortunately, as the discussion in Section 3 shows, the singularity structure of Borel transformed WKB solutions of the Mathieu equation that is relevant to its two simple poles is now clarified. Using this knowledge for the Mathieu equation, we discuss the singularity structure of Borel transformed WKB solutions of an M2P1T equation in what follows.

We first relate the ∞-Mathieu equation with the Mathieu equation. To this end we use the following relation:

$$\tilde{\psi}^{\dagger}_{\pm}(x, a, \rho, \eta) = \tilde{\psi}_{\pm}(x, a, A(a, \rho, \eta), B(a, \rho, \eta), \eta). \tag{4.14}$$

Applying the Borel transformation to (4.14), we can relate the Borel transform $\tilde{\psi}^{\dagger}_{\pm,B}$ of $\tilde{\psi}^{\dagger}_{\pm}$ with $\tilde{\psi}_{\pm,B}$ through the action of a microdifferential operator

$$\mathcal{AB} =: \exp\left(\tilde{A}\theta_A + \tilde{B}\theta_B\right) :, \tag{4.15}$$

where

$$\tilde{A}(a, \rho, \eta) = A(a, \rho, \eta) - A_0(a, \rho), \tag{4.16}$$

$$\tilde{B}(a, \rho, \eta) = B(a, \rho, \eta) - B_0(a, \rho) \tag{4.17}$$

and θ_A (resp. θ_B) is the symbol of ∂_A (resp. ∂_B). Thanks to the estimates (1.3.12) and (1.3.13) in [Part I, Theorem 1.3.1], we obtain the following

Proposition 4.2. *Let $\tilde{\psi}^{\dagger}_{\pm,B}$ and $\tilde{\psi}_{\pm,B}$ respectively be Borel transformed WKB solutions of the ∞-Mathieu equation and the Mathieu equation that are normalized at their simple poles $x = a$. Then they satisfy the following relation:*

$$\begin{aligned} &\tilde{\psi}^{\dagger}_{\pm,B}(x, a, \rho, y) \\ &= \int_{\mp y_+}^{y} K_{A,B}(a, \rho, y - y', \partial_A, \partial_B)\tilde{\psi}_{\pm,B}(x, a, A, B, y')dy' \Big|_{\substack{A = A_0(a,\rho) \\ B = B_0(a,\rho)}}, \end{aligned} \tag{4.18}$$

where $K_{A,B}(a, \rho, y - y', \partial_A, \partial_B)$ is a differential operator of infinite order that is defined on

$$\{(a, \rho, A, B, y) \in \mathbb{C}^5 : 0 < |\rho| < r, R_0|a| < |\rho|, R_1|y| < \sqrt{|\rho|}\} \tag{4.19}$$

with some positive constants r, R_0 and R_1 and

$$y_+(x, a, \rho) = \int_a^x \sqrt{\frac{aA(a, \rho) + xB(a, \rho)}{x^2 - a^2}}dx. \tag{4.20}$$

Now we study the singularity structure of $\tilde{\psi}^{\dagger}_{\pm,B}$ using Theorem 3.2. Let us focus on the m-th singular point of $\tilde{\psi}_{\pm,B}$ located at (3.81). Evidently, from (3.41), the following condition should be satisfied:

$$2\sqrt{2}|m|\pi h(|B_0(a,\rho)/A_0(a,\rho)|) < 1, \tag{4.21}$$

where $h(\delta)$ is a function that satisfies (3.28) $\sim$ (3.30). Since $A_0(0,0) = f^{(1)}(0,0) \neq 0$ and $B_0(0,\rho) = \rho$, [Part I, Lemma 1.2.3] tells us that, by taking R_0 sufficiently large, we can assume that $A_0(a,\rho)$ and $B_0(a,\rho)$ satisfy

$$\frac{1}{2}\big|f^{(1)}(0,0)\big| \leq \big|A_0(a,\rho)\big| \leq \frac{3}{2}\big|f^{(1)}(0,0)\big|, \tag{4.22}$$

$$\frac{1}{2}|\rho| \leq \big|B_0(a,\rho)\big| \leq \frac{3}{2}|\rho| \tag{4.23}$$

on $\{R_0|a| < |\rho|\}$. Since $h(\delta)$ is an increasing function, we find that (4.21) follows from

$$2\sqrt{2}|m|\pi h(3|\rho|/|f^{(1)}(0,0)|) < 1. \tag{4.24}$$

Therefore, by taking ρ sufficiently small with keeping the relation $R_0|a| < |\rho|$, we can make $|B_0(a,\rho)/A_0(a,\rho)|$ arbitrary small so that (4.24) holds. On the other hand, when

$$2|m|\pi R_1\sqrt{|a\Gamma_0(a,A_0(a,\rho),B_0(a,\rho))|} < \sqrt{|\rho|} \tag{4.25}$$

is satisfied, the m-th singular point is contained in the domain of definition of the integro-differential operator in (4.18). Hence, in view of (3.40) and (4.22), it suffices to take a sufficiently small relative to ρ so that

$$2\sqrt{3}|m|\pi R_1\sqrt{|f^{(1)}(0,0)|}\sqrt{|a|} < \sqrt{|\rho|} \tag{4.26}$$

holds. Then, using Theorem 3.2 and Proposition 4.2, we can show the following

Lemma 4.1. *Let $\tilde{\psi}^{\dagger}_{\pm}$ denote the WKB solutions of the ∞-Mathieu equation (4.2) that are normalized at a simple pole $x = a$. Then, when (4.24) and (4.26) hold, its Borel transform $\tilde{\psi}^{\dagger}_{\pm,B}(x,a,A,B,y)$ is singular at*

$$y = \mp y_+(x,a,\rho) + mp(a,\rho) \tag{4.27}$$

and its alien derivative there satisfies

$$\begin{aligned}(4.28)\qquad &\Big(\Delta_{y=\mp y_+ + mp}\ \tilde{\psi}_\pm^\dagger\Big)_B(x,a,\rho,y)\\ &= \pm(-1)^m \Xi_m(\mu,\nu)\big(\exp(-m\oint_\gamma \tilde{S}_{\mathrm{odd}}^\dagger dx)\tilde{\psi}_\pm^\dagger\big)_B(x,a,\rho,y),\end{aligned}$$

where

$$(4.29)\qquad p(a,\rho) = \int_\gamma \sqrt{\frac{f(t,a,\rho)}{t^2-a^2}}dt$$

and $\Xi_m(\mu,\nu)$, $\mu(a)$ *and* $\nu(a)$ *are functions that are given by* (3.83), (3.84) *and* (3.85) *respectively.*

Proof. We first note the following relation, which is an immediate consequence of Proposition 3.1:

$$\begin{aligned}(4.30)\qquad &\oint_\gamma \tilde{S}_{\mathrm{odd}}^\dagger(x,a,\rho,\eta)dx\\ &= 2\pi i\sqrt{a}\Lambda(a,\Gamma(a,A(a,\rho,\eta),B(a,\rho,\eta),\eta),\eta)\eta + \pi i,\end{aligned}$$

where Λ and Γ are formal power series given in Section 1, $\tilde{S}_{\mathrm{odd}}^\dagger$ is the odd part of a solution of the Riccati equation associated with the ∞-Mathieu equation and γ is a contour that encircles two simple poles of the ∞-Mathieu equation counterclockwise avoiding its simple turning point. Especially, we find

$$(4.31)\qquad p(a,\rho) = 2\pi i\sqrt{a\Gamma_0(a,A_0(a,\rho),B_0(a,\rho))}.$$

Then, in a way parallel to the proof of Lemma 3.3, applying Lemma 3.2 to the symbol of $\mathcal{AB}$ and using a coordinate transformation $F:(y,A,B)\to(y',A',B')$ defined by

$$(4.32)\qquad \begin{cases} y' = y - 2m\pi i\sqrt{a\Gamma_0(a,A,B)}\\ A' = A\\ B' = B\end{cases}$$

instead of (3.45), we obtain Lemma 4.1.

□

Remark 4.1. From (3.40), (4.22) and (4.31), we find

$$|p(a,\rho)| = O(\sqrt{|a|}) \tag{4.33}$$

when a tends to 0.

Now, from [Part I, Theorem 1.3.2] and (4.30), we obtain the following

Proposition 4.3. *Let $\hat{S}_{\mathrm{odd}}$ be the odd part of a solution of the Riccati equation associated with an M2P1T equation. Then*

$$\begin{aligned}\oint_\gamma & \hat{S}_{\mathrm{odd}}(t,a,\rho,\eta)dt \\ &= 2\pi i\sqrt{a}\Lambda(a,\Gamma(a,A(a,\rho,\eta),B(a,\rho,\eta),\eta),\eta)\eta + \pi i\end{aligned} \tag{4.34}$$

holds, where γ is a contour that encircles two simple poles of the M2P1T equation counterclockwise avoiding its simple turning point.

In conclusion, combining Proposition 4.1, Lemma 4.1 and Proposition 4.3, we obtain

Theorem 4.2. *Let $\hat{\psi}_\pm(t,a,\rho,\eta)$ be WKB solutions of a generic (i.e. $a \neq 0$, $\rho \neq 0$) M2P1T equation that is normalized at a simple pole $t = a$. Then, for each integer m we can take some positive constants δ_1 and δ_2 so that the following holds when $|\rho| < \delta_1$ and $0 < |a| < \delta_2|\rho|$ are satisfied: The Borel transform $\hat{\psi}_{\pm,B}(t,a,\rho,y)$ of $\hat{\psi}_\pm(t,a,\rho,\eta)$ is singular at*

$$y = \mp y_+(t,a,\rho) + mp(a,\rho) \tag{4.35}$$

and its alien derivative there satisfies

$$\begin{aligned}&\left(\Delta_{y=\mp y_+ + mp}\ \hat{\psi}_\pm\right)_B(t,a,\rho,y) \\ &= \pm(-1)^m \Xi_m(\mu,\nu)\big(\exp(-m\oint_\gamma \tilde{S}_{\mathrm{odd}}dx)\hat{\psi}_\pm\big)_B(t,a,\rho,y),\end{aligned} \tag{4.36}$$

where

$$\Xi_m(\mu,\nu) = \frac{1}{m}\left\{1 + (-1)^m - \cosh\left(2\pi i m\sqrt{\frac{\mu^2+\sqrt{\mu^4-\nu^2}}{2}}\right)\right. \tag{4.37}$$

$$- \cosh\left(2\pi i m\sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}}\right)\Bigg\},$$

$$\mu = \mu(a) = \sqrt{1 + 2(g_+(a) + g_-(-a))}, \tag{4.38}$$

$$\nu = \nu(a) = 2(g_+(a) - g_-(-a)), \tag{4.39}$$

$$y_+(t, a, \rho) = \int_a^t \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}}\,dt \tag{4.40}$$

and

$$p(a, \rho) = \int_\gamma \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}}\,dt. \tag{4.41}$$

Here γ is a contour that encircles two simple poles of the M2P1T equation counterclockwise avoiding its simple turning point.

Remark 4.2. In Theorem 4.2, the positive constants δ_1 and δ_2 should be taken so small that (4.24) and (4.26) are satisfied for $|\rho| < \delta_1$ and $0 < |a| < \delta_2|\rho|$ for an arbitrarily given $m \in \mathbb{Z}$.

Remark 4.3. In (t, a, ρ)-coordinate, $y_+(t, a, \rho)$ is given by (4.40). However, since $x_0(t, a, \rho)$ satisfies

$$\frac{f(t, a, \rho}{t^2 - a^2} = \left(\frac{\partial x_0}{\partial t}\right)^2 \frac{aA_0 + x_0 B_0}{x_0^2 - a^2} \tag{4.42}$$

we find (4.40) is equivalent to (4.13).

References

[Part I] S. Kamimoto, T. Kawai and Y. Takei: Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point I —– its WKB-theoretic transformation to the Mathieu equation.

[A1] T. Aoki: Symbols and formal symbols of pseudodifferential operators, Advanced Studies in Pure Mathematics, **4**, Kinokuniya, 1984, pp.181–208.

[A2] ———: Calcul exponentiel des opérateurs microdifférentiels d'ordre infini. I, Ann. Inst. Fourier, Grenoble **33** (1983), 227-250.

[AKY] T. Aoki, K. Kataoka and S. Yamazaki: Hyperfunctions, FBI transformations and pseudo-differential operators of infinite order, (in Japanese) Kyoritsu-Shuppan CO., LTD, 2004.

[AKT] T. Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory. II, Advanced Studies in Pure Mathematics, **54**, Math. Soc. Japan, 2009, pp.19–94.

[AY] T. Aoki and J. Yoshida: Microlocal reduction of ordinary differential operators with a large parameter, Publ. RIMS, Kyoto Univ., **29**(1993), 959–975.

[DP] E. Delabaere and F. Pham: Resurgent methods in semi-classical asymptotics, Ann. Inst. Henri Poincaré, **71**(1999), 1–94.

[Ec] J. Ecalle: Les fonctions résurgentes, I, II, III, Publ. Math. d'Orsay, Univ. Paris-Sud, 1981 (Tome I, II), 1985 (Tome III).

[Er] A. Erdélyi: Higher Transcendental Functions, I, II, III, McGraw-Hill, 1955; reprinted in 1981 by Robert E. Krieger Publishing Company, Malabar, Florida.

[KKT] S. Kamimoto, T. Kawai and Y. Takei: Microlocal analysis of fixed singularities of WKB solutions of a Schrödinger equation with a merging triplet of two simple poles and a simple turning point, to appear in The Mathematical Legacy of Leon Ehrenpreis, 1930 - 2010.

[K] T. Kawai: Systems of linear differential equations of infinite order — an aspect of infinite analysis, Proc. Symp. in Pure Math., **49**, Part 1, Amer. Math. Soc., 1989, pp.3–17.

[KKoT] T. Kawai, T. Koike and Y. Takei: On the exact WKB analysis of higher order simple-pole type operators, Adv. in Math., **228** (2011), 63–96.

[KT] T. Kawai and Y. Takei: Algebraic Analysis of Singular Perturbation Theory, Amer. Math. Soc., 2005.

[Ko] T. Koike: Voros coefficient of the Legendre equation (private communication).

[KoS] T. Koike and R. Schäfke: in preparation.

[KoT] T. Koike and Y. Takei: On the Voros coefficient for the Whittaker equation with a large parameter —– Some progress around Sato's conjecture in exact WKB analysis, Publ. RIMS, Kyoto Univ., **47** (2011), 375–395.

[P] F. Pham: Resurgence, quantized canonical transformations, and multi-instanton expansion, Algebraic Analysis, Vol. II, Academic Press, 1988, pp.699–726.

[SKK] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp.265–529.

[Sa] D. Sauzin: Resurgent functions and splitting problems, RIMS Kôkyûroku, **1493**, 2006, pp.48–117.

[SS] H. Shen and H. J. Silverstone: Observations on the JWKB treatment of the quadratic barrier, Algebraic Analysis of Differential Equations, Springer, 2008, pp.307–319.

[T] Y. Takei: Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points, RIMS Kôkyûroku Bessatsu, **B10** (2008), 205–224.